Meijer G Function Representations

Kelly Roach
Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

Abstract

An algorithm for computing formula representations of instances of the Meijer G function is discussed. This algorithm is a generalization of an algorithm from a previous paper by the same author. The current paper discusses the Meijer G function briefly; the theory, strategy, and lookup routine certificates of the new algorithm; and applications to the problem of definite integration.

1 Introduction

Our previous paper "Hypergeometric Function Representations" [15], presented an algorithm for computing formula representations of the hypergeometric function F defined by

$$\mathrm{F}\left(\vec{a};\vec{b};z\right)=\sum_{j=0}^{\infty}\frac{\left(\vec{a}\right)_j}{\left(\vec{b}\right)_j}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\,,\,\begin{pmatrix}\vec{a}+j,\vec{b}\\ \vec{a},\vec{b}+j,1+j\end{pmatrix}z^j$$

where we use notation

$$(a)_j=a\ (a+1)\ (a+2)\ \dots\ (a+j-1)$$
$$(a_1,\dots,a_m)_j=(a_1)_j\ \dots\ (a_m)_j$$
$$,\begin{pmatrix}a_1,\dots,a_m\\ b_1,\dots,b_n\end{pmatrix}=\frac{\prod_{i=1}^{m}\,,\ (a_i)}{\prod_{i=1}^{n}\,,\ (b_i)}$$

For example,

$$\mathrm{F}\left(-\frac{3}{2},-\frac{1}{2};\frac{1}{2};z\right)=\frac{2+z}{2}\sqrt{1-z}+\frac{3\sqrt{z}}{2}\sin^{-1}\left(\sqrt{z}\right)$$

is a typical formula representation. Abilitiy to compute such representations is applicable to integration, differential equations, closed form summation, and difference equations [7], [10], [13].

The Meijer G function, $\mathrm{G}(\vec{a};\vec{b};\vec{c};\vec{d};z)$, defined in the next section, is a generalization of the hypergeometric function $\mathrm{F}(\vec{a};\vec{b};z)$. Every hypergeometric function is a G function:

$$\mathrm{F}\left(\vec{a};\vec{b};z\right)=,\begin{pmatrix}\vec{b}\\ \vec{a}\end{pmatrix}\mathrm{G}\left(1-\vec{a};;0;1-\vec{b};\log\left(-z\right)\right)$$

However, not every G function has a simple representation in terms of hypergeometric functions. In particular, Bessel functions Y_μ and K_μ ($\mu\in\mathbb{Z}$), Kelvin functions ker_μ and kei_μ ($\mu\in\mathbb{Z}$), Whittaker function $\mathrm{W}_{\mu\nu}$ ($\nu\in\frac{1}{2}\mathbb{Z}$), Lommel function $\mathrm{S}_{\mu\nu}$ ($\nu\in\mathbb{Z}$), and Legendre function Q_μ^ν ($\nu\in\mathbb{Z}$) can only be represented by G functions.

Our new algorithm computes formula representations such as

$$\begin{aligned}&\mathrm{G}\left(1;;\frac{\mu}{2}+\frac{n}{2}+\frac{1}{2};0,-\frac{\mu}{2}+\frac{n}{2}+\frac{1}{2};2\log\left(\frac{z}{2}\right)\right)\\ &\quad=\frac{z^{n+1}}{\mu+n+1}J_\mu(z)+z\,J_{\mu+1}(z)\,\mathrm{s}_{\mathrm{n},\mu}(z)\\ &\qquad-\frac{z}{\mu+n+1}J_\mu(z)\,\mathrm{s}_{\mathrm{n}+1,\mu+1}(z)\end{aligned}$$

An ability to produce such representations is crucially important to the solution of hypergeometric type integrals which appear copiously in various integral tables [5], [11], [12], [13], used by scientists and mathematicians.

In this paper, we repeat some familiar themes from our previous work [15], **shift operators**, **contiguity relations**, **inverse shift operators**, **suitable origins**, **accessible** origins, **proper sequences**, and lookup **certificates** but in a new and different context. Just the same, the current paper is completely self-contained and will stand on its own.

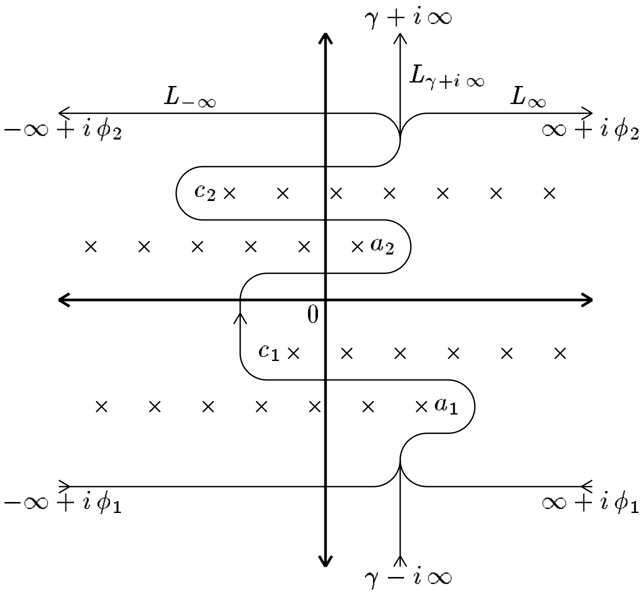
2 Definition

We define the **Meijer G function** by the inverse Laplace transform

$$\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right) = \frac{1}{2\,\pi\,i}\oint_L\,,\,\begin{pmatrix}1-\vec{a}+y,\vec{c}-y\\ \vec{b}-y,1-\vec{d}+y\end{pmatrix}\,e^{y\,z}\,dy$$

where L is one of three types of integration paths $L_{\gamma+i\,\infty}$, L_{∞}, and $L_{-\infty}$.

A schematic plot of the integration path L (L_{∞}, $L_{-\infty}$, or $L_{\gamma+i\,\infty}$) and the poles of the integrand ($\times$) is shown below.



Contour L is one of three types of integration paths L_{∞}, $L_{-\infty}$, and $L_{\gamma+i\,\infty}$. Contour L_{∞} starts at $\infty+i\,\phi_1$ and finishes at $\infty+i\,\phi_2$. Contour $L_{-\infty}$ starts at $-\infty+i\,\phi_1$ and finishes at $-\infty+i\,\phi_2$. Contour $L_{\gamma+i\,\infty}$ starts at $\gamma-i\,\infty$ and finishes at $\gamma+i\,\infty$. All the paths L_{∞}, $L_{-\infty}$, and $L_{\gamma+i\,\infty}$ put all c_j+k poles on the right and all other poles of the integrand (which must be of the form $-1+a_j+k$) on the left. Define G_{∞}, $G_{-\infty}$, and $G_{\gamma+i\,\infty}$ to be the G functions defined by the L_{∞}, $L_{-\infty}$, and $L_{\gamma+i\,\infty}$ contours.

Related to this definition of Meijer G, we also define quantities m, n, p, q, β, δ, and σ by $m=|\,\vec{a}\,|$, $n=|\,\vec{b}\,|$, $p=|\,\vec{c}\,|$, $q=|\,\vec{d}\,|$, $\beta=m-n+p-q$, $\delta=m+n-p-q$, and

$$\sigma=\sum_{i=1}^{m}a_i+\sum_{i=1}^{n}b_i-\sum_{i=1}^{p}c_i-\sum_{i=1}^{q}d_i$$

Analysis of the absolute convergence of the contour integral using Stirling's asymptotic formula for the gamma function produces:

Theorem. G_{∞} *converges absolutely if*

(1) $\delta<0$ *or*

(2) $\delta=0$ *and* $\mathrm{Re}(z)<0$ *or*

(3) $\delta=0$, $\mathrm{Re}(z)=0$, *and* $-\mathrm{Re}(\sigma)<-1$

Theorem. $\mathrm{G}_{-\infty}$ *converges absolutely if*

(1) $\delta>0$ *or*

(2) $\delta=0$ *and* $\mathrm{Re}(z)>0$ *or*

(3) $\delta=0$, $\mathrm{Re}(z)=0$, *and* $-\mathrm{Re}(\sigma)<-1$

Theorem. $\mathrm{G}_{\gamma+\mathrm{i}\,\infty}$ *converges absolutely if*

(1) $|\,\mathrm{Im}(z)\,|<\beta\,\frac{\pi}{2}$ *or*

(2) $|\,\mathrm{Im}(z)\,|=\beta\,\frac{\pi}{2}$ *and* $-\mathrm{Re}(\sigma)+\delta\left(\gamma+\frac{1}{2}\right)<-1$

3 Relation to Traditional Notation

The Meijer G function is traditionally defined by an inverse Mellin transform

$$\mathrm{G}^{\mathrm{mn}}_{\mathrm{pq}}\left(z\left|\begin{matrix}\vec{a},\vec{b}\\ \vec{c},\vec{d}\end{matrix}\right.\right) = \frac{1}{2\,\pi\,i}\oint_L\,,\,\begin{pmatrix}1-\vec{a}+y,\vec{c}-y\\ \vec{b}-y,1-\vec{d}+y\end{pmatrix}\,z^{y}\,dy$$

Hence the traditional definition is related to our definition by

$$\mathrm{G}^{\mathrm{mn}}_{\mathrm{pq}}\left(z\left|\begin{matrix}a_1,\ldots,a_n,a_{n+1},\ldots,a_p\\ b_1,\ldots,b_m,b_{m+1},\ldots,b_q\end{matrix}\right.\right)$$
$$=\mathrm{G}\Big(a_1,...,a_n;a_{n+1},...,a_p;b_1,...,b_m;b_{m+1},...,b_q;\log(z)\Big)$$

The new notation has some advantages over the old notation. First, the parameters of the Meijer G function are separated out into four natural groups $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$. Second, possibly more controversial, we place $e^{y\,z}$ instead of z^y inside the integrand. We deem this desirable because of the "multi-valued" character of z^y. Finally, the $^{\mathrm{mn}}_{\mathrm{pq}}$ subscripts and superscripts which are now redundant are omitted.

4 Properties

The Meijer G function has various properties [4], [6], [13]. Among those of interest us are:

Theorem. (Basic Properties.)

$$\mathrm{G}\left(\mu,\vec{a};\vec{b};\vec{c};\mu,\vec{d};z\right)=\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)$$

$$\mathrm{G}\left(\vec{a};\mu,\vec{b};\mu,\vec{c};\vec{d};z\right)=\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)$$

$$\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};-z\right)=\mathrm{G}\left(1-\vec{c};1-\vec{d};1-\vec{a};1-\vec{b};z\right)$$

$$e^{t\,z}\,\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)=\mathrm{G}\left(\vec{a}+t;\vec{b}+t;\vec{c}+t;\vec{d}+t;z\right)$$

Theorem. (Duplication Formula.)

$$\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};\frac{z}{k}\right) = (2\pi)^{-(k-1)\beta/2}\, k^{1+\delta/2-\sigma} \times \mathrm{G}\Big(\Delta(\vec{a},k);\Delta\left(\vec{b},k\right);\Delta(\vec{c},k);\Delta\left(\vec{d},k\right); z+k\,\delta\log(k)\Big)$$

where we use notation

$$\Delta(\vec{a},k) = \frac{\vec{a}}{k},\frac{\vec{a}+1}{k},\frac{\vec{a}+2}{k},\ldots,\frac{\vec{a}+k-1}{k}$$

Theorem. (Slater's Theorem.) *If G_∞ converges and the elements of $\vec{c}$ are distinct mod 1, then*

$$\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right) = \sum_{h=1}^{p}\Bigg(,\ \begin{pmatrix}1-\vec{a}+c_h, c^*-c_h\\ \vec{b}-c_h, 1-\vec{d}-c_h\end{pmatrix} e^{c_h z} \times \mathrm{F}\Big(1-\vec{a}+c_h, 1-\vec{b}+c_h; 1-c^*+c_h, 1-\vec{d}+c_h; (-1)^{n-p}\, e^z\Big)\Bigg)$$

where $c^* = \vec{c}$ with c_h omitted.

5 Integration Theorems

Four theorems below are not original but serve as a small reference guide to the reader indicating the usefulness of the Meijer G function to solving integration problems. These theorems are very general since many special functions can be represented as G functions. We omit some rather complicated technical conditions on parameters which appear in the last three theorems pertaining to definite integration. Readers may consult section 2.24 of *Integrals and Series Volume 3: More Special Functions* [13] for their complete statement and additional theorems.

Theorem. (Indefinite Integration.)

$$\int \mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)\,dz = \mathrm{G}\left(1,\vec{a};\vec{b};\vec{c};0,\vec{d};z\right)$$

Theorem. (One G Function.)

$$\int_0^\infty z^t\,\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};u\log(z)+v\right)\,dz = \frac{1}{u}\,e^{-\alpha v},\ \begin{pmatrix}-\vec{a}+1-\alpha, \vec{c}+\alpha\\ \vec{b}+\alpha, -\vec{d}+1-\alpha\end{pmatrix}$$

where

$$\alpha = \frac{t+1}{u}$$

Theorem. (Two G Functions.)

$$\int_0^\infty z^t\,G_1\,G_2\,dz = \frac{1}{u}\,e^{-\alpha v_2}\,G_3$$

where

$$G_1 = \mathrm{G}\left(\vec{a_1};\vec{b_1};\vec{c_1};\vec{d_1};u\log(z)+v_1\right)$$

$$G_2 = \mathrm{G}\left(\vec{a_2};\vec{b_2};\vec{c_2};\vec{d_2};u\log(z)+v_2\right)$$

$$G_3 = \mathrm{G}\Big(\vec{a}_1, -\vec{c_2}-\alpha+1; \vec{b}_1, -\vec{d_2}-\alpha+1; \vec{c_1}, -\vec{a}_2-\alpha+1; \vec{d_1}, -\vec{b}_2-\alpha+1; v_1-v_2\Big)$$

$$\alpha = \frac{t+1}{u}$$

Theorem. (Cauchy Principal Value Integral.)

$$⨍_0^\infty \frac{\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};\log(z)+v\right)}{z-\mu}\,dz = -\pi\,\mathrm{G}\left(0,\vec{a};-\frac{1}{2},\vec{b};0,\vec{c};-\frac{1}{2},\vec{d};v+\log(\mu)\right)$$

6 Shift Operators

Define the shift operators A_i, B_i, C_i, and D_i by

$$A_i = D + (-a_i+1)$$

$$B_i = -D + (b_i-1)$$

$$C_i = -D + c_i$$

$$D_i = D - d_i$$

where $D = (\partial/\partial z)$ is the operator for differentiation. It can be seen that A_i and B_i decrement indices and that C_i and D_i increment indices. Visibly,

$$A_i\,\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right) = \mathrm{G}\left(\vec{a}-\vec{e}_i;\vec{b};\vec{c};\vec{d};z\right)$$

$$B_i\,\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right) = \mathrm{G}\left(\vec{a};\vec{b}-\vec{e}_i;\vec{c};\vec{d};z\right)$$

$$C_i\,\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right) = \mathrm{G}\left(\vec{a};\vec{b};\vec{c}+\vec{e}_i;\vec{d};z\right)$$

$$D_i\,\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right) = \mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d}+\vec{e}_i;z\right)$$

where $\vec{e}_i$ are unit vectors.

7 Differential Equation

Applying products of shift operators to $\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)$ we see that

$$\left(\prod_{i=1}^{m} A_i \prod_{i=1}^{n} B_i\right)\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right) = \mathrm{G}\left(\vec{a}-1;\vec{b}-1;\vec{c};\vec{d};z\right)$$

$$\left(\prod_{i=1}^{p} C_i \prod_{i=1}^{q} D_i\right)\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right) = \mathrm{G}\left(\vec{a};\vec{b};\vec{c}+1;\vec{d}+1;z\right)$$

It can be checked that

$$e^z\,\mathrm{G}\left(\vec{a}-1;\vec{b}-1;\vec{c};\vec{d};z\right)=\mathrm{G}\left(\vec{a};\vec{b};\vec{c}+1;\vec{d}+1;z\right)$$

Hence,

$$\left(e^z\prod_{i=1}^{m}A_i\prod_{i=1}^{n}B_i-\prod_{i=1}^{p}C_i\prod_{i=1}^{q}D_i\right)\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)=0$$

Converting to D notation, we get the differential equation for $\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)$. If we let L_1, L_2, and L be the operators

$$L_1=(-1)^{n+p}\,e^z\prod_{j=1}^{m}\left(D+(-a_j+1)\right)\prod_{j=1}^{n}\left(D+(1-b_j)\right)$$

$$L_2=\prod_{j=1}^{p}\left(D-c_j\right)\prod_{j=1}^{q}\left(D-d_j\right)$$

$$L=L_1-L_2$$

then the differential equation for $\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)$ can be written

$$L\,\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)=0$$

8 Contiguity Relations

Operator L is a polynomial in D but

$$D+\mu=A_i+(\mu+a_i-1)$$

$$D+\mu=-B_i+(\mu+b_i-1)$$

$$D+\mu=-C_i+(\mu+c_i)$$

$$D+\mu=D_i+(\mu+d_i)$$

so L can also be expressed as a polynomial in terms of shift operators A_i, B_i, C_i, and D_i converting the differential equation for $\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)$ into a difference equation among contiguous instances of G which we call a contiguity relation.

Let X stand for A, B, C, or D and χ stand for α, β, γ, or δ respectively. If we express L as a polynomial in X_i, then we get

$$L_1=(\pm)\,e^z\,X_i^{m+n}+\ldots+0$$

$$L_2=(\pm)\,X_i^{p+q}+\ldots+\chi_0\left(\vec{a},\vec{b},\vec{c},\vec{d},z\right)$$

$$L=\chi_{\mathrm{d}}\left(\vec{a},\vec{b},\vec{c},\vec{d},z\right)X_i^d+\ldots+\chi_0\left(\vec{a},\vec{b},\vec{c},\vec{d},z\right)$$

where the $\pm$ signs depend on m, n, p, q and whether X is A, B, C, D and $d=\max(m+n,p+q)$.

These results let us define

$$A_i^{-1}=-\sum_{j=0}^{d-1}\frac{\alpha_{\mathrm{j+1}}\left(\vec{a}+\vec{e}_i,\vec{b},\vec{c},\vec{d},z\right)}{\alpha_0\left(\vec{a}+\vec{e}_i,\vec{b},\vec{c},\vec{d},z\right)}A_i^j$$

$$B_i^{-1}=-\sum_{j=0}^{d-1}\frac{\beta_{\mathrm{j+1}}\left(\vec{a},\vec{b}+\vec{e}_i,\vec{c},\vec{d},z\right)}{\beta_0\left(\vec{a},\vec{b}+\vec{e}_i,\vec{c},\vec{d},z\right)}B_i^j$$

$$C_i^{-1}=-\sum_{j=0}^{d-1}\frac{\gamma_{\mathrm{j+1}}\left(\vec{a},\vec{b},\vec{c}-\vec{e}_i,\vec{d},z\right)}{\gamma_0\left(\vec{a},\vec{b},\vec{c}-\vec{e}_i,\vec{d},z\right)}C_i^j$$

$$D_i^{-1}=-\sum_{j=0}^{d-1}\frac{\delta_{\mathrm{j+1}}\left(\vec{a},\vec{b},\vec{c},\vec{d}-\vec{e}_i,z\right)}{\delta_0\left(\vec{a},\vec{b},\vec{c},\vec{d}-\vec{e}_i,z\right)}D_i^j$$

The coefficients of these polynomials in A_i, B_i, C_i, and D_i are defined when

$$\alpha_0\left(\vec{a}+\vec{e}_i,\vec{b},\vec{c},\vec{d},z\right)=-\prod_{j=1}^{p}\left(-c_j+a_i\right)\prod_{j=1}^{q}\left(-d_j+a_i\right)\neq 0$$

$$\beta_0\left(\vec{a},\vec{b}+\vec{e}_i,\vec{c},\vec{d},z\right)=-\prod_{j=1}^{p}\left(-c_j+b_i\right)\prod_{j=1}^{q}\left(-d_j+b_i\right)\neq 0$$

$$\gamma_0\left(\vec{a},\vec{b},\vec{c}-\vec{e}_i,\vec{d},z\right)=(-1)^{m+p}\prod_{j=1}^{m}\left(-a_j+c_i\right)\prod_{j=1}^{n}\left(-b_j+c_i\right)e^z\neq 0$$

$$\delta_0\left(\vec{a},\vec{b},\vec{c},\vec{d}-\vec{e}_i,z\right)=(-1)^{n+p}\prod_{j=1}^{m}\left(-a_j+d_i\right)\prod_{j=1}^{n}\left(-b_j+d_i\right)e^z\neq 0$$

Operators

$$A_i^n=\left(D-a_i+n\right)\ldots\left(D-a_i+1\right)$$

$$B_i^n=(-1)^n\left(D-b_i+n\right)\ldots\left(D-b_i+1\right)$$

$$C_i^n=(-1)^n\left(D-c_i-n-1\right)\ldots\left(D-c_i\right)$$

$$D_i^n=\left(D-d_i-n-1\right)\ldots\left(D-d_i\right)$$

are defined for all a_i, b_i, c_i, and d_i.

9 Contiguity Relations II

For example, using the ideas of the previous section, our routine Contig computes the following contiguity relation:

$$
\begin{aligned}
&\mathrm{G}\left(a_1+1;;c_1,c_2;d_1;z\right)\\
&=-\frac{1}{\left(a_1-d_1\right)\left(a_1-c_2\right)\left(a_1-c_1\right)}\,\mathrm{G}\left(a_1-2;;c_1,c_2;d_1;z\right)\\
&\quad-\frac{3\,a_1-c_1-c_2-d_1-3}{\left(a_1-d_1\right)\left(a_1-c_2\right)\left(a_1-c_1\right)}\,\mathrm{G}\left(a_1-1;;c_1,c_2;d_1;z\right)\\
&\quad-\Big(3\,a_1^2-2\,a_1\,c_1-2\,a_1\,c_2-2\,a_1\,d_1+c_1\,c_2+c_1\,d_1\\
&\qquad+c_2\,d_1-e^z-3\,a_1+c_1+c_2+d_1+1\Big)\\
&\qquad\times\left(a_1-d_1\right)^{-1}\left(a_1-c_2\right)^{-1}\left(a_1-c_1\right)^{-1}\\
&\qquad\times\mathrm{G}\left(a_1;;c_1,c_2;d_1;z\right)
\end{aligned}
$$

10 Proper Sequences and Suitable Origins

Definition A sequence $\vec{S}$ of shift and inverse shift operators A_i, B_i, C_i, D_i, A_i^{-1}, B_i^{-1}, C_i^{-1}, and D_i^{-1} is a **proper sequence** if the composition $S_{|S|}\ldots S_1$ is defined.

Definition A quadruple $(\vec{a_0};\vec{b_0};\vec{c_0};\vec{d_0})$ is a **suitable origin** if $\{\vec{a_0},\vec{b_0}\}$ and $\{\vec{c_0},\vec{d_0}\}$ are disjoint. (Hence, $a_{0i}\neq c_{0i}$, $a_{0i}\neq d_{0i}$, $b_{0i}\neq c_{0i}$, and $b_{0i}\neq d_{0i}$.)

Definition A quadruple $(\vec{a};\vec{b};\vec{c};\vec{d})$ is **accessible** from a quadruple $(\vec{a_0};\vec{b_0};\vec{c_0};\vec{d_0})$ if there exists a constant $t\in\mathbb{C}$ and a proper sequence $\vec{S}$ of shift and inverse shift operators A_i, B_i, C_i, D_i, A_i^{-1}, B_i^{-1}, C_i^{-1}, and D_i^{-1} such that

$$
\begin{aligned}
&\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)\\
&\quad=S_{|S|}\ldots S_1\,\mathrm{G}\left(\vec{a_0}+t;\vec{b_0}+t;\vec{c_0}+t;\vec{d_0}+t;z\right)
\end{aligned}
$$

11 Strategy

Assume $\{\vec{a},\vec{b}\}$ and $\{\vec{c},\vec{d}\}$ are disjoint. Suppose $t\in\mathbb{C}$ and $\vec{k}=\vec{a_0}+t-\vec{a}$, $\vec{l}=\vec{b_0}+t-\vec{b}$, $\vec{m}=\vec{c}-\vec{c_0}-t$, $\vec{n}=\vec{d}-\vec{d_0}-t\in\mathbb{Z}$. We would try

$$
\begin{aligned}
&\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)\\
&\quad=\prod_{i=1}^{m}A_i^{k_i}\prod_{i=1}^{n}B_i^{l_i}\prod_{i=1}^{p}C_i^{m_i}\prod_{i=1}^{q}D_i^{m_i}\,e^{t\,z}\\
&\qquad\times\mathrm{G}\left(\vec{a_0};\vec{b_0};\vec{c_0};\vec{d_0};z\right)
\end{aligned}
$$

but this will not always work because of restrictions on where A_i^{-1}, B_i^{-1}, C_i^{-1}, and D_i^{-1} are defined.

Given any vector $\vec{v}$, let $[\vec{v}]_r$ be the subvector of elements of $\vec{v}$ which are congruent to r mod 1. Given any permutation π of $\{1,\ldots,|\vec{v}|\}$ let $\pi(\vec{v})=(v_{\pi(1)},\ldots,v_{\pi(|\vec{v}|)})$.

Let

$$
\vec{x}=(a_1,\ldots,a_m,b_1,\ldots,b_n,c_1,\ldots,c_p,d_1,\ldots,d_q)
$$

Let π be a permutation which sorts $\vec{x}$ into nondescending order. Let $\vec{y}=\pi(\vec{x})$. Then $[\vec{v}]_r$ is nondescending for every $r\in[0,1)$.

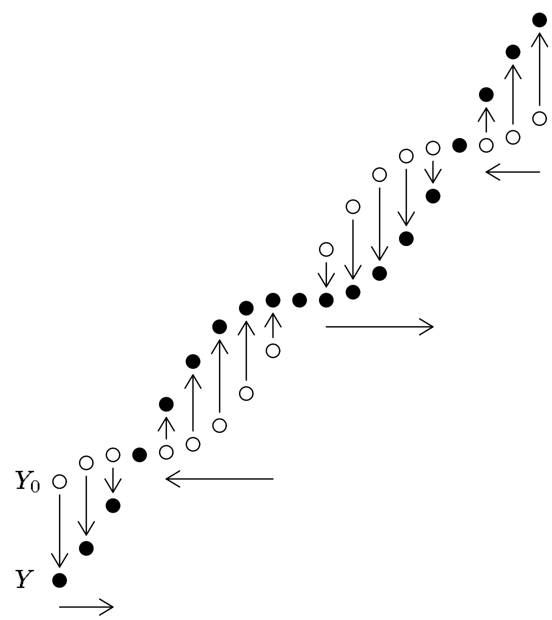
Assume $(a_0;b_0;c_0;d_0)$ is a suitable origin such that $t\in\mathbb{C}$ and $\vec{k}=\vec{a_0}+t-\vec{a}$, $\vec{l}=\vec{b_0}+t-\vec{b}$, $\vec{m}=\vec{c}-\vec{c_0}-t$, $\vec{n}=\vec{d}-\vec{d_0}-t\in\mathbb{Z}$. Let

$$
\vec{x}_0=(a_{01},...,a_{0m},b_{01},...,b_{0n},c_{01},...,c_{0p},d_{01},...,d_{0q})
$$

$$
\vec{X}=(A_1^{k_1},...,A_m^{k_m},B_1^{l_1},...,B_n^{l_n},C_1^{m_1},...,C_p^{m_p},D_1^{n_1},...,D_q^{n_q})
$$

Let $\vec{y}_0=\pi(\vec{x}_0)$ and $\vec{Y}=\pi(\vec{X})$. Assume $[\vec{y}_0]_r$ is nondescending for every $r\in[0,1)$.

For any given $r\in[0,1)$, plot the elements of $[\vec{y}]_r$ and $[\vec{y}_0]_r$ as a function of position. Call the resulting monotonic polygonal curves Y and Y_0. For example, we might get this picture:



To avoid $\{\vec{a},\vec{b}\}$ and $\{\vec{c},\vec{d}\}$ having elements in common as we apply X_i operators to $e^{t\,z}\,\mathrm{G}(\vec{a_0};\vec{b_0};\vec{c_0};\vec{d_0})$ we may proceed left to right where Y lies below Y_0 and right to left where Y lies above or on Y_0.

Let ϕ be a permutation of $\vec{y}$ that in every plot of $[\vec{y}]_r$ and $[\vec{y}_0]_r$ for every $r\in[0,1)$ selects the elements of $[\vec{y}]_r$ from left to right where Y lies below Y_0 and selects the elements of $[\vec{y}]_r$ from right to left where Y lies above or on Y_0. Then we should apply X_i operators to $e^{t\,z}\,\mathrm{G}(\vec{a_0};\vec{b_0};\vec{c_0};\vec{d_0})$ in the order $X_{\phi(\pi(1))},\ldots,X_{\phi(\pi(m+n+p+q))}$. That is,

$$
\begin{aligned}
&\mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};z\right)\\
&\quad=X_{\phi(\pi(m+n+p+q))}\ldots X_{\phi(\pi(1))}\,e^{t\,z}\\
&\qquad\times\mathrm{G}\left(\vec{a_0};\vec{b_0};\vec{c_0};\vec{d_0};z\right)
\end{aligned}
$$

12 Formula Algorithm

proc Formula($\vec{a},\vec{b},\vec{c},\vec{d}$)
$\vec{a}$:=sort($\vec{a}$)
$\vec{b}$:=sort($\vec{b}$)
$\vec{c}$:=sort($\vec{c}$)
$\vec{d}$:=sort($\vec{d}$)
Delete elements $\vec{a}$ and $\vec{d}$ have in common.
Delete elements $\vec{b}$ and $\vec{c}$ have in common.
$[\vec{a}_0,\vec{b}_0,\vec{c}_0,\vec{d}_0,B,C,M,\rho]$:=Lookup($\vec{a},\vec{b},\vec{c},\vec{d}$);
$[\vec{a}_0,\vec{b}_0,\vec{c}_0,\vec{d}_0,plan]$:=Plan($\vec{a},\vec{b},\vec{c},\vec{d},\vec{a}_0,\vec{b}_0,\vec{c}_0,\vec{d}_0$);
for bucket **in** plan **do**
  $[shift,e]$:=$bucket$;
  if $e<0$ **then**
    for j **from** 1 **to** $-e$ **do**
      $[\vec{a}_0,\vec{b}_0,\vec{c}_0,\vec{d}_0,C]$:=
        Unshift($shift,\vec{a}_0,\vec{b}_0,\vec{c}_0,\vec{d}_0,z^\rho,C,M$);
    od;
  elif $e>0$ **then**
    for j **from** 1 to e **do**
      $[\vec{a}_0,\vec{b}_0,\vec{c}_0,\vec{d}_0,C]$:=
        Shift($shift,\vec{a}_0,\vec{b}_0,\vec{c}_0,\vec{d}_0,z^\rho,C,M$);
    od;
  fi;
od;
return subs($z=z^{1/\rho},C\cdot B$);

13 Lookup Routine

The **Lookup** routine currently consists of 48 procedures each of which, in effect, add infinitely many $[\vec{a}_0,\vec{b}_0,\vec{c}_0,\vec{d}_0,B,C,M,\rho]$ certificates to a virtual lookup table.

The following table summarizes the number of formulas in **Lookup** by their (m,n,p,q) classification:

$(0,0,2,0)$	1	$(0,0,2,2)$	1
$(0,0,3,1)$	2	$(0,1,2,0)$	2
$(0,1,2,1)$	3	$(0,1,2,3)$	1
$(0,1,3,0)$	1	$(0,1,3,2)$	1
$(0,1,4,1)$	2	$(0,2,3,1)$	2
$(0,2,4,0)$	1	$(0,3,4,1)$	1
$(1,0,1,2)$	3	$(1,0,2,0)$	2
$(1,0,2,1)$	6	$(1,0,3,0)$	3
$(1,1,2,1)$	1	$(1,1,2,2)$	3
$(1,1,3,1)$	2	$(1,1,4,0)$	1
$(2,0,2,2)$	2	$(2,0,3,1)$	2
$(2,1,2,3)$	2		

14 Results and Conclusion

Due to their complexity and lack of space, we will not present a number of more advanced theorems related to calculation of Meijer G Function Representations. We just say that these theorems go by names such as **Paired Index Theorems** (similar to theorems in Adamchik [3]), a **PFD Duplication Formula** (related to a similar formula in Roach [15]), and an **Expansion Theorem** (a generalization of Slater's Theorem).

One of our long term goals is to enlarge our **Lookup** routine to the point that our algorithm should basically reproduce nearly all 879 (roughly) of the formula representations for the Meijer G function listed in chapter 8 of *Integrals and Series Volume 3: More Special Functions* [13]. We are not at that point yet, but progress is good. Every formula in this book through our algorithm turns into infinitely many formulas. We also envision that our algorithm will appear as an important subroutine inside general routines which solve integration problems.

In the course of this work, we discovered mistakes in formulas 2(19), 12(7), 12(8), 15(7), 15(8), 20(8), 20(45), 22(15), 22(16), 22(21), 22(22), 23(34), 25(5), 29(15), 29(16), 40(6), 40(22), 40(23), 49(41), 49(42), 49(44), 18(15), 18(16), 43(1), 43(2), 46(9), and 46(10) of section 8.4 of *Integrals and Series Volume 3: More Special Functions* [13]. We have not inspected sections 41 and 42 discussing the Legendre functions P_μ^ν and Q_μ^ν closely enough yet to comment about their correctness, but otherwise this list of errors may be nearly comprehensive.

15 Gallery

The following integrals, most of which appear in *Integrals and Series* [11], [12], [13] were calculated with the aid of the theorems and algorithm described in this paper. The performance of two different computer algebra systems on this test suite is as follows: Maple 5.4 was able to compute a formula for one integral and left all the other integrals unevaluated. Mathematica 2.2 left six integrals unevaluated, produced four answers which still contained hypergeometric functions F, and only computed formulas for three of these integrals.

$$\int z^n\,J_\mu(z)\,dz$$
$$=\frac{z^{n+1}}{\mu+n+1}\,J_\mu(z)+z\,J_{\mu+1}(z)\,\mathrm{s}_{n,\mu}(z)$$
$$-\frac{z}{\mu+n+1}\,J_\mu(z)\,\mathrm{s}_{n+1,\mu+1}(z)$$

(2.5.6(3) p390 v1)

$$\int_0^\infty \frac{\sin(b\,x)}{(x^2+z^2)^\rho}\,dx$$

$$= -\frac{\csc(\pi\,\rho)\; z^{1/2-\rho}\, b^{\rho-1/2}\, 2^{1/2-\rho}\, \pi^{3/2}}{2\,,\;(\rho)}\, I_{\frac{2\,\rho+3}{2}}(b\,z)$$

$$- \frac{2^{-1/2-\rho}\,\sqrt{\pi}\, z^{1/2-\rho}\, b^{\rho-1/2}\,,\;(-\rho+2)}{\rho-1}\,\mathbf{L}_{-\frac{2\,\rho-1}{2}}(b\,z)$$

$$- \frac{\pi^{3/2}\, b^{\rho-3/2}\, 2^{1/2-\rho}\, z^{-1/2-\rho}\;(2\,\rho+1)\,\csc(\pi\,\rho)}{2\,,\;(\rho)}\, I_{\frac{2\,\rho+1}{2}}(b\,z)$$

(2.5.21(3b) p430 v1)

$$\int_0^\infty \cos\left(a\,x^2+2\,b\,x\right)\,dx$$

$$= \frac{\sqrt{\pi}\,\sqrt{2}\cos\left(\frac{b^2}{a}\right)}{4\,\sqrt{a}} + \frac{\sqrt{\pi}\,\sqrt{2}\sin\left(\frac{b^2}{a}\right)}{4\,\sqrt{a}}$$

$$- \frac{\sqrt{\pi}\,\sqrt{2}\cos\left(\frac{b^2}{a}\right)}{2\,\sqrt{a}}\,\mathrm{C}\left(\frac{\sqrt{2}\,b}{\sqrt{\pi}\,\sqrt{a}}\right)$$

$$- \frac{\sqrt{\pi}\,\sqrt{2}\sin\left(\frac{b^2}{a}\right)}{2\,\sqrt{a}}\,\mathrm{S}\left(\frac{\sqrt{2}\,b}{\sqrt{\pi}\,\sqrt{a}}\right)$$

(2.7.6(6) p560 v1)

$$\int_0^\infty \cos(b\,x)\,\tan^{-1}\left(\frac{a}{x^2}\right)\,dx$$

$$= \frac{\pi\sin\left(\frac{\sqrt{2}\,b\,\sqrt{a}}{2}\right)}{b}\exp\left(-\frac{\sqrt{2}\,b\,\sqrt{a}}{2}\right)$$

(2.12.19(6) p200 v2)

$$\int_0^\infty \frac{\cos(b\,\sqrt{x})\;J_0(c\,x)}{\sqrt{x}}\,dx$$

$$= \frac{b\,\pi}{4\,c}\,J_{\frac{7}{4}}\left(\frac{b^2}{8\,c}\right)^2 + \frac{36\,\pi\,c}{b^3}\,J_{\frac{3}{4}}\left(\frac{b^2}{8\,c}\right)^2$$

$$- \frac{b\,\pi}{4\,c}\,J_{\frac{1}{4}}\left(\frac{b^2}{8\,c}\right)^2 - \frac{6\,\pi}{b}\,J_{\frac{3}{4}}\left(\frac{b^2}{8\,c}\right)\,J_{\frac{7}{4}}\left(\frac{b^2}{8\,c}\right)$$

(2.12.31(1l) p210 v2)

$$\int_0^\infty \frac{J_1(b\,x)\;J_1(c\,x)}{x^2}\,dx$$

$$= \frac{2\,b^2-2\,c^2}{3\,b\,\pi}\,\mathrm{K}\left(\frac{b}{c}\right) + \frac{2\,b^2+2\,c^2}{3\,b\,\pi}\,\mathrm{E}\left(\frac{b}{c}\right)$$

(2.12.31(1u) p210 v2)

$$\int_0^\infty x^{3/2}\,J_0(b\,x)\;J_0(c\,x)\;dx$$

$$= \frac{1}{2\,\sqrt{b}\,,\;\left(\frac{3}{4}\right)^2\,(b^2-c^2)}\,\mathrm{K}\left(\frac{\sqrt{2-2\,\sqrt{\frac{b^2-c^2}{b^2}}}}{2}\right)$$

$$+ \frac{1}{b^{5/2}\,,\;\left(\frac{3}{4}\right)^2\,\left(\frac{b^2-c^2}{b^2}\right)^{3/2}}\,\mathrm{K}\left(\frac{\sqrt{2-2\,\sqrt{\frac{b^2-c^2}{b^2}}}}{2}\right)$$

$$- \frac{2}{b^{5/2}\,,\;\left(\frac{3}{4}\right)^2\,\left(\frac{b^2-c^2}{b^2}\right)^{3/2}}\,\mathrm{E}\left(\frac{\sqrt{2-2\,\sqrt{\frac{b^2-c^2}{b^2}}}}{2}\right)$$

(2.14.1(6a) p290 v2)

$$\int_0^\infty e^{i\,p\,x}\,\mathrm{H}_0^{(1)}(c\,x)\;dx$$

$$= \frac{1}{c\,\sqrt{\frac{c^2-p^2}{c^2}}} - \frac{2}{\pi\,c\,\sqrt{\frac{c^2-p^2}{c^2}}}\,\sin^{-1}\left(\frac{p}{c}\right)$$

(2.15.20(4c) p320 v2)

$$\int_0^\infty e^{-p\,x}\,I_1(c\,x)^2\;dx$$

$$= -\frac{p}{\pi\,c^2}\,\mathrm{E}\left(\frac{2\,c}{p}\right) - \frac{2\,c^2-p^2}{p\,\pi\,c^2}\,\mathrm{K}\left(\frac{2\,c}{p}\right)$$

(2.15.20(5f) p320 v2)

$$\int_0^\infty \frac{e^{-p\,x}\,I_1(c\,x)^2}{x}\,dx$$

$$= -\frac{1}{2} + \frac{p^2}{2\,\pi\,c^2}\,\mathrm{E}\left(\frac{2\,c}{p}\right) + \frac{4\,c^2-p^2}{2\,\pi\,c^2}\,\mathrm{K}\left(\frac{2\,c}{p}\right)$$

(2.16.15(1a) p360 v2)

$$\int_0^\infty x^{\nu+1}\sin\left(\frac{c\,x^2}{2\,a}\right)\;K_\nu(c\,x)\;dx$$

$$= \frac{2^\nu\,c^{-\nu-1}\,a\;(\nu-1)\;,\;(\nu-1)}{2}$$

$$+ \frac{\pi\,a^{\nu+1}\sec\left(\frac{\pi\,\nu}{2}\right)\;\sin\left(\frac{a\,c}{2}\right)}{4\,c}$$

$$- \frac{\pi\,a^{\nu+1}\csc\left(\frac{\pi\,\nu}{2}\right)\;\cos\left(\frac{a\,c}{2}\right)}{4\,c}$$

$$+ \frac{,\;(\nu-1)\;a^{\nu+1/2}\,\sqrt{2}}{2\,c^{3/2}}\,\mathrm{s}_{-\frac{2\,\nu-3}{2},\frac{1}{2}}\left(\frac{a\,c}{2}\right)$$

$$- \frac{,\;(\nu-1)\;\nu\,a^{\nu+3/2}\,\sqrt{2}}{4\,\sqrt{c}}\,\mathrm{s}_{-\frac{2\,\nu-1}{2},\frac{3}{2}}\left(\frac{a\,c}{2}\right)$$

(2.16.15(2a) p360 v2)

$$\int_0^\infty \sin\left(b\,x^2\right)\,K_\nu\,(c\,x)\,dx$$
$$= -\frac{\pi^{3/2}\csc\left(\frac{\pi\,(\nu+1)}{4}\right)\,\sin\left(\frac{c^2}{8\,b}\right)\,\csc\left(\frac{\pi\,\nu}{2}\right)}{16\,\sqrt{b}}\,J_{\frac{\nu}{2}}\left(\frac{c^2}{8\,b}\right)$$
$$-\frac{\pi^{3/2}\sec\left(\frac{\pi\,(\nu+1)}{4}\right)\,\cos\left(\frac{c^2}{8\,b}\right)\,\csc\left(\frac{\pi\,\nu}{2}\right)}{16\,\sqrt{b}}\,J_{\frac{\nu}{2}}\left(\frac{c^2}{8\,b}\right)$$
$$+\frac{\pi^{3/2}\csc\left(-\frac{\pi\,(\nu-1)}{4}\right)\,\sin\left(\frac{c^2}{8\,b}\right)\,\csc\left(\frac{\pi\,\nu}{2}\right)}{16\,\sqrt{b}}\,J_{-\frac{\nu}{2}}\left(\frac{c^2}{8\,b}\right)$$
$$+\frac{\pi^{3/2}\sec\left(-\frac{\pi\,(\nu-1)}{4}\right)\,\cos\left(\frac{c^2}{8\,b}\right)\,\csc\left(\frac{\pi\,\nu}{2}\right)}{16\,\sqrt{b}}\,J_{-\frac{\nu}{2}}\left(\frac{c^2}{8\,b}\right)$$

(2.7.16(3) p90 v3)

$$\int_0^\infty x\,J_{-\nu}\,(b\,x)\,\left(Y_\nu\,(c\,x)-\mathbf{H}_\nu\,(c\,x)\right)\,dx$$
$$= -\frac{2\,b^{-\nu-2}\,c^\nu\,b^2\cos\left(\pi\,\nu\right)}{\pi\,\left(b^2-c^2\right)}+\frac{2\,b^{-3-\nu}\,c^{\nu+1}\,b^2\cos\left(\pi\,\nu\right)}{\pi\,\left(b^2-c^2\right)}$$

References

[1] Abramowitz, M. and Stegun I. A. (eds.) (1965), *Handbook of Mathematical Functions*, Dover Publications, Inc., New York.

[2] Adamchik, V. and Marichev O. I. (1990), "The Algorithm for Calculating Integrals of Hypergeometric Type Functions and its Realization in REDUCE System", *Proceedings of ISSAC '90*, 301–308. ACM, New York.

[3] Adamchik, V. (1995), "The Evaluation of Integrals of Bessel Functions via G-Function Identities", *J. Comp. and Applied Math*, **64**, 283-290.

[4] Erdelyi, A. (ed.) (1953), *Higher Transcendental Functions, Volume I*, Robert E. Krieger Publishing Company, Malabar, Florida.

[5] Gradshteyn, I. S. and Ryzhik, I. M. (1965), *Table of Integrals, Series, and Products*, Academic Press.

[6] Luke, Y. L. (1969), *The Special Functions and Their Approximations, Volume I*, Academic Press, San Diego.

[7] Marichev, O. I. (1983), *Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables*, Ellis Horwood Limited, Chichester, England.

[8] Mathai, A. M. (1993), *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Oxford University Press, Oxford.

[9] McPhedran, R. C., Dawes, D. H., and Scott, T. (1992), "On a Bessel Function Integral", *The Maple Technical Newsletter*, 33-38, Birkhäuser, Boston.

[10] Petkovšek, M. and Salvy, B. (1993), "Finding All hypergeometric Solutions of Linear Differential Equations", *Proceedings of ISSAC '93*, ACM, New York.

[11] Prudnikov, A. P., Brychkov, Yu. A., Marichev O. I. (1990), *Integrals and Series, Volume 1: Elementary Functions*, Gordon and Breach Science Publishers.

[12] Prudnikov, A. P., Brychkov, Yu. A., Marichev O. I. (1990), *Integrals and Series, Volume 2: Special Functions*, Gordon and Breach Science Publishers.

[13] Prudnikov, A. P., Brychkov, Yu. A., Marichev O. I. (1990), *Integrals and Series, Volume 3: More Special Functions*, Gordon and Breach Science Publishers.

[14] Prudnikov, A. P., Brychkov, Yu. A., Marichev O. I. (1991), "Evaluation of Integrals and the Mellin Transform", *Journal of Soviet Mathematics*, **54(6)**, (translated from Russian original in *Itogi Nauki i Tekhniki, Seriya Matmaticheskii Analiz*, **27**, 3-146).,

[15] Roach, K. (1996), "Hypergeometric Function Representations", *Proceedings of ISSAC '96*, 301–308. ACM, New York.

[16] Slater, L. (1966), "Generalized Hypergeometric Functions", Cambridge University Press.