# Hypergeometric Function Representations

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**Abstract**

An algorithm for computing formula representations of instances of the generalized hypergeometric function is presented. Examples of lookup routine certificates and the underlying theory of the algorithm are discussed in the paper. Finally, a gallery of results are presented and the algorithm is compared to already existing routines in Macsyma, Maple, and Mathematica.

## 1 Introduction

The hypergeometric function F can be defined by

$$\mathrm{F}\left(\vec{a};\vec{b};z\right) = \sum_{j=0}^{\infty} \frac{(\vec{a})_j}{\left(\vec{b}\right)_j} \frac{z^j}{j!} = \sum_{j=0}^{\infty} , \left(\begin{matrix} \vec{a}+j, \vec{b} \\ \vec{a}, \vec{b}+j, 1+j \end{matrix}\right) z^j$$

where $\vec{a}$ and $\vec{b}$ are vectors, $p = |\vec{a}|$, $q = |\vec{b}|$, $p \leq q+1$, and $|z| < 1$ if $p = q+1$. Our objective is to compute representations for instances of $F$. For example,

$$\mathrm{F}\left(-\frac{3}{2}, -\frac{1}{2}; \frac{1}{2}; z\right) = \frac{2+z}{2}\sqrt{1-z} + \frac{3\sqrt{z}}{2}\sin^{-1}\left(\sqrt{z}\right)$$

Various simple expressions and well known functions can be expressed in term of F. These include exponentials, binomials, logarithms, trigonometric functions, inverse trigonometric functions, incomplete Gamma function, error function, Fresnel integrals, Bessel functions, Kelvin functions, Airy functions, Struve functions, Anger J function, Weber E function, Whittaker functions, complete elliptic integrals, orthogonal polynomials, Lommel functions, polylogarithms, and Lerch $\Phi$ function [1], [7]. For example,

$$\sin^{-1}(z) = z\,\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right)$$

$$\mathrm{L}_\mu(z) = \frac{z^{\mu+1}}{2^\mu \sqrt{\pi}\, ,\, \left(\frac{3}{2}+\mu\right)} \mathrm{F}\left(1; \frac{3}{2}, \frac{3}{2}+\mu; \frac{z^2}{4}\right)$$



$$\mathrm{s}_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)\,(\mu+\nu+1)} \times \mathrm{F}\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right)$$

hypergeometric functions are applicable to integration, differential equations, closed form summation, and difference equations [5], [6], [7]. Some methods will create answers in terms of F. An algorithm like ours can often reexpress such answers in terms of better known functions.

## 2 ${}_2F_1$ Example

Let ${}_pF_q$ denote the restriction of F to $\mathbb{C}^p \times \mathbb{C}^q \times \mathbb{C}$. Then ${}_0F_0$, ${}_0F_1$, and ${}_1F_0$ are representable by

$$\mathrm{F}(;;z) = e^z$$

$$\mathrm{F}(;b;z) = \,,\,(b)\, z^{\frac{-b+1}{2}} \mathrm{I}_{b-1}\left(2\sqrt{z}\right)$$

$$\mathrm{F}(a;;z) = (1-z)^{-a}$$

presenting no further challenge to us.

We proceed to a ${}_2F_1$ example which is more interesting. Let $D = (\partial/\partial z)$ be the operator for differentiation. The following shift relations are known:

$$\mathrm{F}(a_1+1, a_2; b_1; z) = \left(\frac{z}{a_1} D + 1\right) \mathrm{F}(a_1, a_2; b_1; z)$$

$$\mathrm{F}(a_1, a_2; b_1-1; z) = \left(\frac{z}{b_1-1} D + 1\right) \mathrm{F}(a_1, a_2; b_1; z)$$

Also, the following contiguity relations are known:

$$\mathrm{F}(a_1-1, a_2; b_1; z) = -\frac{(a_1-a_2)\,z - 2\,a_1 + b_1}{a_1-b_1} \mathrm{F}(a_1, a_2; b_1; z) - \frac{a_1\,(1-z)}{a_1-b_1} \mathrm{F}(a_1+1, a_2; b_1; z)$$

$$\mathrm{F}(a_1, a_2; b_1+1; z) = -\frac{b_1\,((a_1+a_2-2\,b_1+1)\,z + b_1 - 1)}{z\,(a_1-b_1)\,(a_2-b_1)} \times \mathrm{F}(a_1, a_2; b_1; z) + \frac{b_1\,(b_1-1)\,(1-z)}{z\,(a_1-b_1)\,(a_2-b_1)} \mathrm{F}(a_1, a_2; b_1-1; z)$$

Using these shift and contiguity relations, we can start from almost any $\mathrm{F}(a_1, a_2; b_1; z)$ representation to obtain any $\mathrm{F}(a_1+m_1, a_2+m_2; b_1+n_1; z)$ representation where $m_1$, $m_2$, $n_1 \in \mathbb{Z}$. The denominators appearing in the shift relations and contiguity relations are troublesome since we can't let them become zero.

We show how to compute $\mathrm{F}\left(-\frac{3}{2}, -\frac{1}{2}; \frac{1}{2}; z\right)$ by starting from the known formula for $\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right)$.

Known:

$$\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right) = \frac{\sin^{-1}\left(\sqrt{z}\right)}{\sqrt{z}}$$

Shift:

$$\mathrm{F}\left(\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; z\right) = (2\,z\,D+1)\;\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right) = \frac{1}{\sqrt{1-z}}$$

Contiguity:

$$\begin{aligned}\mathrm{F}&\left(-\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right)\\ &= \frac{1}{2}\,\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right) + \frac{1-z}{2}\,\mathrm{F}\left(\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; z\right)\\ &= \frac{1}{2}\sqrt{1-z} + \frac{1}{2\sqrt{z}}\sin^{-1}\left(\sqrt{z}\right)\end{aligned}$$

Contiguity:

$$\begin{aligned}\mathrm{F}&\left(-\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; z\right)\\ &= \frac{5-2\,z}{4}\,\mathrm{F}\left(-\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right) - \frac{1-z}{4}\,\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right)\\ &= \frac{5-2\,z}{8}\sqrt{1-z} + \frac{3}{8\sqrt{z}}\sin^{-1}\left(\sqrt{z}\right)\end{aligned}$$

Shift:

$$\begin{aligned}\mathrm{F}&\left(-\frac{3}{2}, \frac{3}{2}; \frac{3}{2}; z\right) = (2\,z\,D+1)\;\mathrm{F}\left(-\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; z\right)\\ &= (1-z)^{3/2}\end{aligned}$$

Contiguity:

$$\begin{aligned}\mathrm{F}&\left(-\frac{3}{2}, -\frac{1}{2}; \frac{3}{2}; z\right)\\ &= \frac{1+4\,z}{2}\,\mathrm{F}\left(-\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; z\right) + \frac{1-z}{2}\,\mathrm{F}\left(-\frac{3}{2}, \frac{3}{2}; \frac{3}{2}; z\right)\\ &= \frac{13+2\,z}{16}\sqrt{1-z} + \frac{3+12\,z}{16\sqrt{z}}\sin^{-1}\left(\sqrt{z}\right)\end{aligned}$$

Shift:

$$\begin{aligned}\mathrm{F}&\left(-\frac{3}{2}, -\frac{1}{2}; \frac{1}{2}; z\right) = (2\,z\,D+1)\;\mathrm{F}\left(-\frac{3}{2}, -\frac{1}{2}; \frac{3}{2}; z\right)\\ &= \frac{2+z}{2}\sqrt{1-z} + \frac{3\sqrt{z}}{2}\sin^{-1}\left(\sqrt{z}\right)\end{aligned}$$

Hence, we conclude:

$$\mathrm{F}\left(-\frac{3}{2}, -\frac{1}{2}; \frac{1}{2}; z\right) = \frac{2+z}{2}\sqrt{1-z} + \frac{3\sqrt{z}}{2}\sin^{-1}\left(\sqrt{z}\right)$$

This formula is not explicitly listed in [7]. Neither Mathematica 2.2 nor Maple 5.3 will compute it. Macsyma 419.0 does compute it, but returns a wrong answer.

We will develop a strategy using shift relations and contiguity relations for ${}_pF_q$ where $p$ and $q$ are arbitrary (subject to $p \le q+1$) to compute $\mathrm{F}(\vec{a}+\vec{m}; \vec{b}+\vec{n}; z)$ from $\mathrm{F}(\vec{a}; \vec{b}; z)$. First, we will study shift relations and contiguity relations for general ${}_pF_q$.

## 3 Shift Operators

We define shift operators $A_i$ and $B_i$ which apply to expressions $\mathrm{F}(\vec{a}; \vec{b}; z)$ where $\vec{a}$ and $\vec{b}$ are constant vectors. Let $D = (\partial/\partial z)$ be the operator for differentiation. It can be checked that

$$\left(\frac{z}{\mu}\,D+1\right)\;\mathrm{F}\left(\mu, \vec{a}; \vec{b}; z\right) = \mathrm{F}\left(\mu+1, \vec{a}; \vec{b}; z\right)$$

Define the shift operators $A_i$ by

$$A_i = \frac{z}{a_i}\,D + 1$$

We see that $A_i$ applied to $\mathrm{F}(\vec{a}; \vec{b}; z)$ increments the $i$th upper index of $\mathrm{F}(\vec{a}; \vec{b}; z)$. Similarly, define the shift operator $B_i$ by

$$B_i = \frac{z}{b_i - 1}\,D + 1$$

It can be seen that $B_i$ applied to $\mathrm{F}(\vec{a}; \vec{b}; z)$ decrements the $i$th lower index of $\mathrm{F}(\vec{a}; \vec{b}; z)$. Visibly,

$$A_i\,\mathrm{F}\left(\vec{a}; \vec{b}; z\right) = \mathrm{F}\left(\vec{a}+\vec{e}_i; \vec{b}; z\right)$$

$$B_i\,\mathrm{F}\left(\vec{a}; \vec{b}; z\right) = \mathrm{F}\left(\vec{a}; \vec{b}-\vec{e}_i; z\right)$$

where $\vec{e}_i$ are unit vectors. We need $a_i \neq 0$ and $b_i \neq 1$ for $A_i$ and $B_i$ to be defined.

## 4 Differential Equation

Applying products of shift operators to $F(\vec{a}; \vec{b}; z)$ we see that

$$\left(\prod_{i=1}^{p} A_i\right)\;\mathrm{F}\left(\vec{a}; \vec{b}; z\right) = \mathrm{F}\left(\vec{a}+1; \vec{b}; z\right)$$

$$\left(\prod_{i=1}^{q} B_i\right)\;\mathrm{F}\left(\vec{a}; \vec{b}; z\right) = \mathrm{F}\left(\vec{a}; \vec{b}-1; z\right)$$

It can be checked that

$$(z\,D)\;\mathrm{F}\left(\vec{a}; \vec{b}-1; z\right) = z\,\frac{\prod_{i=1}^{p} a_i}{\prod_{i=1}^{q}(b_i-1)}\,\mathrm{F}\left(\vec{a}+1; \vec{b}; z\right)$$

Hence,

$$\begin{aligned}&\left(z\,\frac{\prod_{i=1}^{p} a_i}{\prod_{i=1}^{q}(b_i-1)}\prod_{i=1}^{p} A_i - (z\,D)\prod_{i=1}^{q} B_i\right)\;\mathrm{F}\left(\vec{a}; \vec{b}; z\right)\\ &\quad = 0\end{aligned}$$

Clearing denominators, we get the differential equation for $\mathrm{F}(\vec{a}; \vec{b}; z)$:

$$\begin{aligned}&\left(z\prod_{i=1}^{p}(z\,D+a_i) - (z\,D)\prod_{i=1}^{q}(z\,D+b_i-1)\right)\\ &\quad \times\mathrm{F}\left(\vec{a}; \vec{b}; z\right)\\ &\quad = 0\end{aligned}$$

In this last equation, no restrictions on $\vec{a}$ and $\vec{b}$ beyond those required by $F(\vec{a}; \vec{b}; z)$ are necessary.

## 5 Contiguity Relations

Let

$$L_1 = z \prod_{j=1}^{p} (z\,D + a_j)$$

$$L_2 = (z\,D) \prod_{j=1}^{q} (z\,D + b_j - 1)$$

$$L = L_1 - L_2$$

The differential equation for $\mathrm{F}\left(\vec{a};\vec{b};z\right)$ becomes

$$L\,\mathrm{F}\left(\vec{a};\vec{b};z\right) = 0$$

Now $L$ is a polynomial in $D$ but

$$z\,D + \mu = a_i\,A_i + (\mu - a_i)$$

$$z\,D + \mu = (b_i - 1)\,B_i + (\mu - b_i + 1)$$

so $L$ can also be expressed as a polynomial in terms of shift operators $A_i$ and $B_i$ converting the differential equation $L\,\mathrm{F}(\vec{a};\vec{b};z) = 0$ into a difference equation among contiguous instances of F which we call a contiguity relation. Operators

$$A_i^n = \left(\frac{z}{a_i + n - 1}\,D + 1\right) \ldots \left(\frac{z}{a_i}\,D + 1\right)$$

$$B_i^n = \left(\frac{z}{b_i - n}\,D + 1\right) \ldots \left(\frac{z}{b_i - 1}\,D + 1\right)$$

are defined if $a_i \notin \{0, -1, \ldots, -n+1\}$ and $b_i \notin \{1, 2, \ldots, n\}$ respectively.

If we express $L$ as a polynomial in $A_i$, then we get

$$L_1 = (a_i)_p\;z\,A_i^p + \ldots + 0$$

$$L_2 = (a_i)_{q+1}\;A_i^{q+1} + \ldots - a_i \prod_{j=1}^{q} (b_j - 1 - a_i)$$

$$L = \ldots - a_i \prod_{j=1}^{q} (b_j - 1 - a_i)$$

$$= \alpha_{\mathrm{d}}\left(\vec{a},\vec{b},z\right)\;A_i^d + \ldots + \alpha_0\left(\vec{a},\vec{b},z\right)$$

which has degree $d = \max(p, q+1)$.

If we express $L$ as a polynomial in $B_i$, we get

$$L_1 = (b_i - p)_p\;z\,B_i^p + \ldots + \left(\prod_{j=1}^{p} (a_j - b_i + 1)\right) z$$

$$L_2 = (b_i - q - 1)_{q+1}\;B_i^{q+1} + \ldots + 0$$

$$L = \ldots + \prod_{j=1}^{q} (a_j - b_i + 1)\;z$$

$$= \beta_{\mathrm{d}}\left(\vec{a},\vec{b},z\right)\;B_i^d + \ldots + \beta_0\left(\vec{a},\vec{b},z\right)$$

which has degree at most $d = \max(p, q+1)$.

These results let us define

$$A_i^{-1} = -\sum_{n=0}^{d-1} \frac{\alpha_{\mathrm{n}+1}\left(\vec{a} - \vec{e}_i, \vec{b}, z\right)}{\alpha_0\left(\vec{a} - \vec{e}_i, \vec{b}, z\right)}\;A_i^n$$

$$B_i^{-1} = -\sum_{n=0}^{d-1} \frac{\beta_{\mathrm{n}+1}\left(\vec{a}, \vec{b} + \vec{e}_i, z\right)}{\beta_0\left(\vec{a}, \vec{b} + \vec{e}_i, z\right)}\;B_i^n$$

The coefficients of these polynomials in $A_i$ and $B_i$ are defined when

$$\alpha_0\left(\vec{a} - \vec{e}_i, \vec{b}, z\right) = -\,(a_i - 1) \prod_{j=1}^{q} (b_j - a_i) \neq 0$$

$$\beta_0\left(\vec{a}, \vec{b} + \vec{e}_i, z\right) = \prod_{j=1}^{p} (a_j - b_i)\;z \neq 0$$

Operators $A_i^n$ and $B_i^n$ for $n = 0, 1, \ldots, d-1$ are defined if $a_i \notin \{0, -1, \ldots, -d+2\}$ and $b_i \notin \{1, 2, \ldots, d-1\}$ respectively. Hence, $A_i^{-1}$ is defined if $a_i \notin \{1, 0, -1, -2, \ldots, -d+2\}$ and $a_i$ is distinct from all $b_j$. $B_i^{-1}$ is defined if $b_i \notin \{1, 2, \ldots, d-1\}$ and $b_i$ is distinct from all $a_j$. Recall that $d = \max(p, q+1)$.

## 6 Contiguity Relations II

Here, for example, are the contiguity relations for ${}_1F_2$ computed by our routine **Contig** which we've implemented using the ideas of the previous section.

$$\begin{aligned}
&\mathrm{F}\left(a_1 - 1; b_1, b_2; z\right) \\
&\quad = \frac{-3\,a_1^2 + 2\,b_1\,a_1 + 2\,b_2\,a_1 - a_1 + z - b_1\,b_2}{(a_1 - b_2)\;(b_1 - a_1)} \\
&\qquad \times \mathrm{F}\left(a_1; b_1, b_2; z\right) \\
&\qquad - \frac{a_1\;(b_1 - 3\,a_1 + b_2 - 2)}{(a_1 - b_2)\;(b_1 - a_1)}\,\mathrm{F}\left(a_1 + 1; b_1, b_2; z\right) \\
&\qquad + \frac{-a_1\;(a_1 + 1)}{(a_1 - b_2)\;(b_1 - a_1)}\,\mathrm{F}\left(a_1 + 2; b_1, b_2; z\right)
\end{aligned}$$

$$\begin{aligned}
&\mathrm{F}\left(a_1; b_1 + 1, b_2; z\right) \\
&\quad = \frac{b_1\;\left(-b_1^2 + b_1 + b_1\,b_2 + z - b_2\right)}{z\;(b_1 - a_1)}\,\mathrm{F}\left(a_1; b_1, b_2; z\right) \\
&\qquad + \frac{b_1\;(b_1 - 1)\;(2\,b_1 - b_2 - 2)}{z\;(b_1 - a_1)}\,\mathrm{F}\left(a_1; b_1 - 1, b_2; z\right) \\
&\qquad - \frac{b_1\;(b_1 - 1)\;(b_1 - 2)}{z\;(b_1 - a_1)}\,\mathrm{F}\left(a_1; b_1 - 2, b_2; z\right)
\end{aligned}$$

## 7 Proper Sequences

Previously, we computed $\mathrm{F}\left(-\frac{3}{2}, -\frac{1}{2}; \frac{1}{2}; z\right)$ by computing $B_1\,A_2^{-1}\,A_1^{-2}\,\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right)$. Could we have computed $A_1^{-2}\,A_2^{-1}\,B_1\,\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right)$ instead? No. Since $A_2^{-1}$ applied to $\mathrm{F}(a_1, a_2; b_1; z)$ is

$$A_2^{-1} = -\frac{(a_2 - a_1)\;z - 2\,a_2 + b_1}{a_2 - b_1} - \frac{a_2\;(1 - z)}{a_2 - b_1}\,A_2$$

the $A_2^{-1}$ which applies to $B_1\,\mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right) = \mathrm{F}\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; z\right)$ would be

$$A_2^{-1} = -\frac{0\,z - 1 + \frac{1}{2}}{\frac{1}{2} - \frac{1}{2}} - \frac{\frac{1}{2}\;(1 - z)}{\frac{1}{2} - \frac{1}{2}}\,A_2$$

which is undefined. The order of the shift and inverse shift operators is therefore important.

**Definition** A sequence $\vec{S}$ of shift and inverse shift operators $A_i$, $B_i$, $A_i^{-1}$, and $B_i^{-1}$ is a **proper sequence** if the composition $S_{|S|} \ldots S_1$ is defined.

## 8 Suitable Origins

Can we compute $\mathrm{F}\left(\frac{1}{2};2;z\right) = B_1^{-1}\,\mathrm{F}\left(\frac{1}{2};1;z\right)$ ? No. Since $B_1^{-1}$ applied to $\mathrm{F}(a_1;b_1;z)$ is

$$B_1^{-1} = -\frac{b_1\ (b_1+z-1)}{z\ (a_1-b_1)} + \frac{b_1\ (b_1-1)}{z\ (a_1-b_1)}\,B_1$$

it would appear that

$$\mathrm{F}\left(\frac{1}{2};2;z\right) = 2\,\mathrm{F}\left(\frac{1}{2};1;z\right) + 0\cdot\mathrm{F}\left(\frac{1}{2};0;z\right)$$

However, $\mathrm{F}\left(\frac{1}{2};0;z\right)$ is undefined. $B_1$ applied to $\mathrm{F}\left(\frac{1}{2};1;z\right)$ would have the form

$$B_1 = \frac{z}{0}\,D + 1$$

which is undefined. Even worse, $\mathrm{F}\left(\frac{1}{2};2;z\right) = 2\,\mathrm{F}\left(\frac{1}{2};1;z\right)$ is false! At $z=0$ this equation gives $1=2$.

Referring back to their definitions, we see $A_i$, $B_i$, $A_i^{-1}$, and $B_i^{-1}$ applied to $\mathrm{F}(\vec{a};\vec{b};z)$ are defined if $a_i \neq 0$, $b_i \neq 1$, $a_i \notin \{1,0,-1,-2,\ldots,-d+2\}$ and $a_i$ is distinct from all $b_j$, and $b_i \notin \{1,2,\ldots,d-1\}$ and $b_i$ is distinct from all $a_j$, respectively. Here, $d = \max(p,q+1)$.

**Definition.** A pair $(\vec{a}_0;\vec{b}_0)$ is a **suitable origin** if

- (1) $\vec{a}_0$ and $\vec{b}_0$ are free of nonpositive integers
- (2) $\vec{a}_0$ and $\vec{b}_0$ are disjoint
- (3) Integer elements of $\vec{b}_0$ are $\geq d = \max(p,q+1)$ ($p = |\vec{a}_0|$ and $q = |\vec{b}_0|$)

**Definition.** A pair $(\vec{a};\vec{b})$ is **accessible** from a pair $(\vec{a}_0;\vec{b}_0)$ if there exists a proper sequence $\vec{S}$ of shift and inverse shift operators $A_i$, $B_i$, $A_i^{-1}$, and $B_i^{-1}$ such that
$\mathrm{F}\left(\vec{a};\vec{b};z\right) = S_{|\,S\,|}\ \ldots\ S_1\,\mathrm{F}\left(\vec{a}_0;\vec{b}_0;z\right)$

## 9 Strategy

Assume $\vec{a}$ and $\vec{b}$ are free of nonpositive integers. Assume $\vec{a}$ and $\vec{b}$ are disjoint. Suppose $\vec{m} = \vec{a}-\vec{a}_0, \vec{n} = \vec{b}_0 - \vec{b} \in \mathbb{Z}$. We would try

$$\mathrm{F}\left(\vec{a};\vec{b};z\right) = \prod_{i=1}^{p} A_i^{m_i} \prod_{i=1}^{q} B_i^{n_i}\,\mathrm{F}\left(\vec{a}_0;\vec{b}_0;z\right)$$

but this will not always work because of restrictions on where $A_i$, $B_i$, $A_i^{-1}$, and $B_i^{-1}$ are defined.

Given any vector $\vec{v}$, let $[\vec{v}]_r$ be the subvector of elements of $\vec{v}$ which are congruent to $r$ mod 1. Given any permutation $\pi$ of $\{1,\ldots,|\vec{v}|\}$ let $\pi(\vec{v}) = (v_{\pi(1)},\ldots,v_{\pi(|\vec{v}|)})$

Let

$$\vec{x} = (a_1,\ldots,a_p,b_1,\ldots,b_q)$$

Let $\pi$ be a permutation which sorts $\vec{x}$ into nondescending order. Let $\vec{y} = \pi(\vec{x})$. Then $[\vec{y}]_r$ is nondescending for every $r \in [0,1)$.

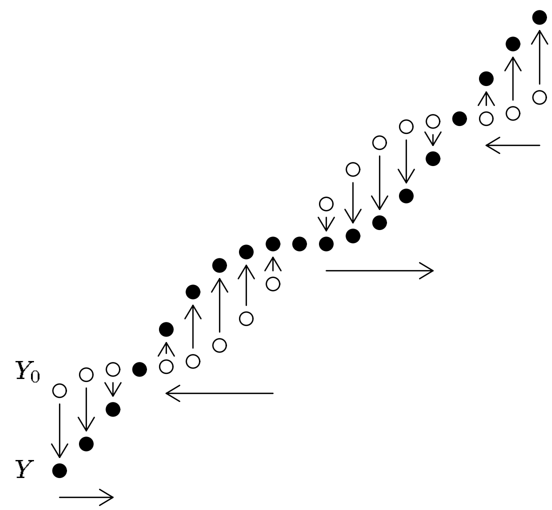
Assume $(a_0;b_0)$ is a suitable origin such that $\vec{m} = \vec{a} - \vec{a}_0, \vec{n} = \vec{b}_0 - \vec{b} \in \mathbb{Z}$. Let

$$\vec{x}_0 = (a_{01},\ldots,a_{0p},b_{01},\ldots,b_{0q})$$

$$\vec{X} = \left(A_1^{m_1},\ldots,A_p^{m_p},B_1^{n_1},\ldots,B_q^{n_q}\right)$$

Let $\vec{y}_0 = \pi(\vec{x}_0)$ and $\vec{Y} = \pi(\vec{X})$. Assume $[\vec{y}_0]_r$ is nondescending for every $r \in [0,1)$.

For any given $r \in [0,1)$, plot the elements of $[\vec{y}]_r$ and $[\vec{y}_0]_r$ as a function of position. Call the resulting monotonic polygonal curves $Y$ and $Y_0$. For example, we might get this picture:




To avoid $\vec{a}$ and $\vec{b}$ having elements in common as we apply $X_i$ operators to $\mathrm{F}(\vec{a}_0;\vec{b}_0;z)$ we may proceed left to right where $Y$ lies below $Y_0$ and right to left where $Y$ lies above or on $Y_0$.

Let $\phi$ be a permutation of $\vec{y}$ that in every plot of $[\vec{y}]_r$ and $[\vec{y}_0]_r$ for every $r \in [0,1)$ selects the elements of $[\vec{y}]_r$ from left to right where $Y$ lies below $Y_0$ and selects the elements of $[\vec{y}]_r$ from right to left where $Y$ lies above or on $Y_0$. Then we should apply $X_i$ operators to $\mathrm{F}(\vec{a}_0;\vec{b}_0;z)$ in the order $X_{\phi(\pi(1))},\ldots,X_{\phi(\pi(p+q))}$. That is,

$$\mathrm{F}\left(\vec{a};\vec{b};z\right) = X_{\phi(\pi(p+q))}\ \cdots\ X_{\phi(\pi(1))}\,\mathrm{F}\left(\vec{a}_0;\vec{b}_0;z\right)$$

## 10 A Theorem

**Theorem.** Let

- (1) $\vec{a}$ and $\vec{b}$ be free of nonpositive integers.
- (2) $\vec{a}$ and $\vec{b}$ be disjoint.
- (3) $\pi$ sort $\vec{x} = (a_1,\ldots,a_p,b_1,\ldots,b_q)$ into nondescending order
- (4) $(\vec{a}_0;\vec{b}_0)$ be a suitable origin
- (5) $\vec{a}-\vec{a}_0, \vec{b}-\vec{b}_0 \in \mathbb{Z}$
- (6) $\vec{x}_0 = (a_{01},\ldots,a_{0p},b_{01},\ldots,b_{0q})$
- (7) $[\pi(\vec{x}_0)]_r$ be nondescending for every $r \in [0,1)$

Then $(\vec{a};\vec{b})$ is accessible from $(\vec{a}_0;\vec{b}_0)$.

## 11 Another Theorem

**Theorem.** The set of hypergeometric functions $\mathrm{F}(\vec{a};\vec{b};z)$ such that $(\vec{a};\vec{b})$ is accessible from an origin $(\vec{a}_0;\vec{b}_0)$ is a subset of a $\mathbb{C}(z)$-module which is generated by a finite basis with size at most $d = \max(p,q+1)$.

**Proof.** This follows from the differential equation for $\mathrm{F}(\vec{a};\vec{b};z)$ which has order $d = \max(p,q+1)$ and the definitions of the shift and inverse shift operators.

## 12 Implementation Specifics

The main routine **Formula**$(\vec{a},\vec{b})$ computes $\mathrm{F}(\vec{a};\vec{b};z)$. There is a subroutine **Lookup** which computes a suitable origin $(\vec{a}_0;\vec{b}_0)$ for $(\vec{a};\vec{b})$. There is a subroutine **Plan** which determines the proper sequence $\vec{S}$ of shift operators and inverse shift operators $A_i$, $B_i$, $A_i^{-1}$, and $B_i^{-1}$ which should be applied to $\mathrm{F}(\vec{a}_0;\vec{b}_0;z)$ to produce $\mathrm{F}(\vec{a};\vec{b};z)$. After the plan is computed, a loop executes the plan by calling subroutines **Shift** and **Unshift**. The **Unshift** routine calls a routine **Contig** which computes contiguity relations and also calls **Shift**.

For aesthetic reasons (i.e. pretty answers) all the routines work, until the very last moment, in terms of a $\mathbb{C}(z)$-module basis $B$. In fact, in the current implementation, $\mathbb{C}(z)$ is always $\mathbb{Q}(z)$, $\rho$ is a positive integer, and the answer will be $\mathrm{F}(\vec{a};\vec{b};z^\rho) = C \cdot B$ after each step of the main loop where $C \in \mathbb{Q}(z)^d$ is a coefficient vector. The basis $B$ is generally some vector of expressions involving special functions. The derivative matrix $M$ which has elements in $\mathbb{Q}(z)$ satisfies the equation $D\,B = M\,B$ where $D = (\partial/\partial z)$. At the last moment, $z$ is replaced by $z^{1/\rho}$ since we are interested in computing $\mathrm{F}(\vec{a};\vec{b};z)$ instead of $\mathrm{F}(\vec{a};\vec{b};z^\rho)$. Often, $\rho = 1$, but not always.

## 13 Main Algorithm

**proc** Formula$(\vec{a},\vec{b})$
  $\vec{a}$:=sort$(\vec{a})$
  $\vec{b}$:=sort$(\vec{b})$
  Delete elements $\vec{a}$ and $\vec{b}$ have in common.
  **if** $\vec{a}$ or $\vec{b}$ contains a nonpositive integer **then**
    **return** polynomial or error
  **else**
    $[\vec{a}_0,\vec{b}_0,B,C,M,\rho]$:=Lookup$(\vec{a},\vec{b})$;
    $[\vec{a}_0,\vec{b}_0,plan]$:=Plan$(\vec{a},\vec{b},\vec{a}_0,\vec{b}_0)$;
    **for** bucket **in** plan **do**
      $[shift,e]$:=$bucket$;
      **if** $e<0$ **then**
        **for** $j$ **from** 1 **to** $-e$ **do**
          $[\vec{a}_0,\vec{b}_0,C]$:=Unshift$(shift,\vec{a}_0,\vec{b}_0,z^\rho,C,M)$;
        **od**;
      **elif** $e>0$ **then**
        **for** $j$ **from** 1 to $e$ **do**
          $[\vec{a}_0,\vec{b}_0,C]$:=Shift$(shift,\vec{a}_0,\vec{b}_0,z^\rho,C,M)$;
        **od**;
      **fi**;
    **od**;
    **return** subs$(z=z^{1/\rho}, C\cdot B)$;
**fi**;

## 14 Lookup Routine

The **Lookup** routine currently consists of:

- A table of 71 different $[\vec{a}_0,\vec{b}_0,B,C,M,\rho]$ entries which we call **certificates**.
- Procedures which implement 19 different formulas, each of which, in effect, add infinitely many more certificates to the lookup table.
- Number of formulas implemented:

| | | | |
|---|---|---|---|
| ${}_0F_0$ | 1 | ${}_0F_1$ | 1 |
| ${}_1F_0$ | 1 | ${}_1F_1$ | 3 |
| ${}_1F_2$ | 3 | ${}_2F_1$ | 4 |
| ${}_0F_3$ | 2 | ${}_0F_q$ | 1 |
| Deriv | 1 | Lerch $\Phi$ | 1 |
| PFD Dupl | 1 | | |

Deriv, Lerch $\Phi$, and PFD Dupl are names of formulas which will be explained later.

## 15 Data

The simplest information available to **Lookup** are the 71 different $[\vec{a}_0,\vec{b}_0,B,C,M,\rho]$ certificates which are stored in a table. This ${}_1F_2$ entry

$$\mathrm{F}\left(1;\frac{1}{2},\frac{1}{2};z\right) = 1+\sqrt{z}\,\pi\,\mathrm{L}_0\left(2\sqrt{z}\right)$$

is stored as a certificate whose components are

$$\vec{a}_0 = [1]$$
$$\vec{b}_0 = \left[\frac{1}{2},\frac{1}{2}\right]$$
$$B = [1, \pi\,\mathrm{L}_0\,(2\,z)\,, \pi\,\mathrm{L}_1\,(2\,z)]$$
$$C = [1, z, 0]$$
$$M = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 2 \\ 0 & 2 & -\frac{1}{z} \end{bmatrix}$$
$$\rho = 2$$

## 16 Small Formulas

We have implemented 16 small formulas for ${}_pF_q$ where $p$ and $q$ are small. For example, this ${}_1F_2$ formula

$$\mathrm{F}\left(a;a+\frac{1}{2},2\,a;z\right) = \,,\ \left(a+\frac{1}{2}\right)^2\,2^{2\,a-1}\,z^{-a+1/2}\,\mathrm{I}_{a-\frac{1}{2}}\left(\sqrt{z}\right)$$

is implemented by a routine which returns a certificate $[\vec{a}_0,\vec{b}_0,B,C,M,\rho]$ whose components are

$$\vec{a}_0 = [a]$$
$$\vec{b}_0 = \left[a+\frac{1}{2}, 2\,a\right]$$
$$B = \Big[\,,\ \left(a+\frac{1}{2}\right)^2 2^{2\,a-1}\,z^{-2\,a}\,\mathrm{I}_{a-\frac{1}{2}}(z)^2,\ ,\ \left(a+\frac{1}{2}\right)^2 2^{2\,a-1}\,z^{-2\,a}\,\mathrm{I}_{a-\frac{1}{2}}(z)\,\mathrm{I}_{a+\frac{1}{2}}(z),\ ,\ \left(a+\frac{1}{2}\right)^2 2^{2\,a-1}\,z^{-2\,a}\,\mathrm{I}_{a+\frac{1}{2}}(z)^2\Big]$$
$$C = [z, 0, 0]$$
$$M = \begin{bmatrix} -\frac{1}{z} & 2 & 0 \\ 1 & -\frac{2\,a+1}{z} & 1 \\ 0 & 2 & -\frac{4\,a+1}{z} \end{bmatrix}$$
$$\rho = 2$$

## 17 Derivative Formula

If an upper index and a lower index differ by a positive integer, we can use differentiation to reduce the order of a hypergeometric function.

**Theorem** *Let* $a - b \in \mathbb{Z}^+$ *and*

$$,\ \begin{pmatrix} a+l, b \\ a, b+l \end{pmatrix} = \sum_i p_i\, l^i \in \mathbb{Q}[l]$$

*Then*

$$\mathrm{F}\left(a, \vec{c}; b, \vec{d}; z\right) = \sum_i p_i\, (z\, D)^i\ \mathrm{F}\left(\vec{c}; \vec{d}; z\right)$$

## 18 Lerch Phi and Polylogarithms

If the coefficients of the series representation of a hypergeometric function are rational functions of the summation index, then the hypergeometric function can be expressed as a linear sum of Lerch $\Phi$ functions. The Lerch $\Phi$ function is defined by

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}$$

Further, if the parameters of the hypergeometric function are rational, we can proceed to express the hypergeometric function as a linear sum of polylogarithms. The polylogarithm function is defined by

$$\mathrm{Li}_\mu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\mu}$$

The first theorem shows how to express such a hypergeometric function as a linear sum of Lerch $\Phi$ functions.

**Theorem** *Let* $\vec{a} - \vec{b} \in \mathbb{Z}$, $c_0 \in \mathbb{Z}^+$, *and*

$$,\ \begin{pmatrix} \vec{a}+l, c_0+l, \vec{b} \\ \vec{a}, c_0, \vec{b}+l, 1+l \end{pmatrix} \in \mathbb{Q}(l)$$

*have partial fraction decomposition*

$$\sum_i p_i\, l^i + \sum_{i,j} \frac{q_{ij}}{(l+r_i)^j}$$

*Then*

$$\mathrm{F}\left(\vec{a}, c_0; \vec{b}; z\right) = \sum_i p_i\ (z\, D)^i\ \frac{1}{1-z} + \sum_{i,j} q_{ij}\ \Phi\left(z, j, r_i\right)$$

The next theorem can be used to range reduce the third argument of a Lerch $\Phi$ into the interval $(0, 1]$.

**Theorem** $\Phi\left(z, s, a+n\right)$

$$= z^{-n} \sum_{k=n}^{-1} \frac{z^k}{(a+k)^s} + z^{-n}\, \Phi(z, s, a) \qquad (n < 0)$$

$$= -z^{-n} \sum_{k=0}^{n-1} \frac{z^k}{(a+k)^s} + z^{-n}\, \Phi(z, s, a) \qquad (n > 0)$$

The next two theorems show how to convert Lerch $\Phi$ into polylogarithms if the third argument is rational.

**Theorem** $\Phi(z, s, 1) = \dfrac{1}{z}\, \mathrm{Li}_s(z)$

**Theorem** *Let* $m \in \{1, \ldots, n\}$ *and* $\zeta_n = e^{2\,\pi\, i/n}$. *Then*

$$\Phi\left(z, s, \frac{m}{n}\right) = n^{s-1} \sum_{k=0}^{n-1} \left(\zeta_n^k\, z^{1/n}\right)^{-m}\ \mathrm{Li}_s\left(\zeta_n^k\, z^{1/n}\right)$$

**Corollary** *Let* $m \in \{1, \ldots, n\}$ *and* $\zeta_n = e^{2\,\pi\, i/n}$. *Then*

$$\Phi\left(z, 1, \frac{m}{n}\right) = -\sum_{k=0}^{n-1} \left(\zeta_n^k\, z^{1/n}\right)^{-m}\ \log\left(1 - \zeta_n^k\, z^{1/n}\right)$$

$$= -z^{-m/n} \log\left(1 - z^{1/n}\right) - \left(\frac{1+(-1)^n}{2}\right)\ (-1)^m\ z^{-m/n} \log\left(1 + z^{1/n}\right) - z^{-m/n} \sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(F_1 - F_2\right)$$

*where*

$$F_1 = \cos\left(\frac{2\,\pi\, k\, m}{n}\right)\ \log\left(1 - 2\cos\left(\frac{2\,\pi\, k}{n}\right)\ z^{1/n} + z^{2/n}\right)$$

$$F_2 = 2\sin\left(\frac{2\,\pi\, k\, m}{n}\right)\ \tan^{-1}\left(\frac{\sin\left(\frac{2\,\pi\, k}{n}\right)\ z^{1/n}}{1 - \cos\left(\frac{2\,\pi\, k}{n}\right)\ z^{1/n}}\right)$$

## 19 PFD Duplication Formula

The most general formula installed as a subroutine of **Lookup** combines the use of partial fraction decomposition and the Gamma duplication formula into a single formula. We use the notation

$$\Delta\left(\vec{a}, n\right) = \frac{\vec{a}}{n}, \frac{\vec{a}+1}{n}, \frac{\vec{a}+2}{n}, \ldots, \frac{\vec{a}+n-1}{n}$$

in the theorem below.

**Theorem** *Let* $\vec{a} - \vec{b} \in \mathbb{Z}$, $c_0 \in \mathbb{Z}^+$, $|\vec{c}| = p$, $|\vec{d}| = q$, $n \in \mathbb{Z}^+$, $\zeta_n = e^{2\,\pi\, i/n}$, *and*

$$,\ \begin{pmatrix} \vec{a}+\frac{l}{n}, c_0+\frac{l}{n}, \vec{b} \\ \vec{a}, c_0, \vec{b}+\frac{l}{n}, 1+\frac{l}{n} \end{pmatrix} \in \mathbb{Q}(l)$$

*have partial fraction decomposition*

$$\sum_i p_i\, l^i + \sum_{i,j} \frac{q_{ij}}{(l+r_i)^j}$$

*Then*

$$\mathrm{F}\left(\vec{a}, c_0, \Delta\left(\vec{c}, n\right); \vec{b}, \Delta\left(\vec{d}, n\right); z\right) = \frac{1}{n} \sum_i p_i \sum_{k=0}^{n-1} (n\, z\, D)^i\ F_1 + \frac{1}{n} \sum_{i,j} q_{ij}\, r_i^{-j} \sum_{k=0}^{n-1} F_2$$

*where*

$$F_1 = \mathrm{F}\left(1, \vec{c}; \vec{d}; n^{-p+q}\, \zeta_n^k\, z^{1/n}\right)$$

$$F_2 = \mathrm{F}\left(1, \vec{c}, \underbrace{r_i, \ldots, r_i}_{j}; \vec{d}, \underbrace{r_i+1, \ldots, r_i+1}_{j}; n^{-p+q}\, \zeta_n^k\, z^{1/n}\right)$$

## 20 Results and Conclusion

The main accomplishment of our algorithm is the essential reproduction of 1504 formulas in 9 tables of representations of $\mathrm{F}(\vec{a};\vec{b};z)$ listed in *Integrals and Series, Volume 3: More Special Functions* [7]. The total number of formulas in each of these tables is neatly summarized by the following table:

| $p,q$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\infty$ 7.11.1 | $\infty$ 7.13.1 | 3 7.16.1 | 11 7.16.2 |
| 1 | $\infty$ 7.3.1 | 72 7.11.2 | 266 7.14.2 | |
| 2 | | 352 7.3.2 | 167 7.12.2 | 7 7.15.2 |
| 3 | | | 621 7.4.2 | |
| 4 | | | | 5 7.5.2 |

The ${}_0F_0$, ${}_0F_1$, and ${}_1F_0$ entries are covered by general formulas. The remaining 9 tables occupy most of the 186 pages of Chapter 7 material on hypergeometric functions. Our algorithm can be used to extend these tables to values of parameters very far out from those given by *Integrals and Series, Volume 3: More Special Functions* [7]. The only limits on distance are the computer resources of time and memory.

The next table indicates the proportion of ${}_pF_q$ formulas with parameters in $\{-\frac{3}{2},-\frac{1}{2},\frac{1}{2},1,\frac{3}{2},2\}$ that can be reduced by our algorithm.

| $p,q$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1.0 | 1.0 | 0 | .57143 |
| 1 | 1.0 | 1.0 | 1.0 | 0 |
| 2 | | 1.0 | .50417 | .56471 |
| 3 | | | .75686 | .28911 |
| 4 | | | | .51186 |

This table means, for example, that our algorithm was able to compute 51.186% of the ${}_4F_3$'s. (Our algorithm does reduce other instances of ${}_0F_2$ and ${}_1F_3$, but none with the parameters mentioned here.)

In more recent work, our algorithm has been extended to compute representations for $\mathrm{F}(\vec{a};\vec{b};-z)$, therefore making our algorithm encompass even more elementary and special functions.

## 21 Gallery

We now present a gallery of formulas produced by our algorithm. While our algorithm has been used to compute thousands of representations for F, we must limit ourselves here to putting on display just a small number of these representations. To make a point of the strength of our algorithm, we've selected examples which are not listed in *Integrals and Series, Volume 3: More Special Functions* [7], cannot be computed by Mathematica 2.2's `HypergeometricPFQ` function, and cannot be computed by Maple 5.3's `hypergeom` function. Macsyma 419.0's `hgfred` function is able to make progress on the first, third, and fourth examples (for the latter two choosing representations in terms of `whittaker_m` and `alegendre_p`) but is unable to eliminate `hyper_f` from the remaining examples. These examples are all quite typical of the formulas that can be produced by our algorithm.

$$\mathrm{F}\left(\frac{1}{2};\frac{9}{2};z\right) = -\frac{525+280\,z+140\,z^2}{128\,z^3}\,e^z + \frac{525+630\,z+420\,z^2+280\,z^3}{256\,z^{7/2}}\,\sqrt{\pi}\,\mathrm{erfi}\left(\sqrt{z}\right)$$

$$\mathrm{F}\left(-\frac{3}{2};-\frac{1}{2},\frac{1}{2};z\right) = (1+2\,z)\cosh\left(2\sqrt{z}\right)+\sqrt{z}\sinh\left(2\sqrt{z}\right) - 4\,z^{3/2}\,\mathrm{Shi}\left(2\sqrt{z}\right)$$

$$\mathrm{F}\left(-\frac{3}{2},-\frac{1}{2};-\frac{5}{2},1;z\right) = \frac{5-4\,z}{5}\,e^{z/2}\,\mathrm{I}_0\left(\frac{z}{2}\right)+\frac{4\,z}{5}\,e^{z/2}\,\mathrm{I}_1\left(\frac{z}{2}\right)$$

$$\mathrm{F}\left(-\frac{3}{2},-\frac{1}{2};2;z\right) = -\frac{4+24\,z-28\,z^2}{15\,z\,\pi}\,\mathrm{K}\left(\sqrt{z}\right) + \frac{4+56\,z+4\,z^2}{15\,z\,\pi}\,\mathrm{E}\left(\sqrt{z}\right)$$

$$\begin{aligned}\mathrm{F}\left(-\frac{1}{2},1;\frac{1}{4},\frac{1}{2},\frac{3}{4};z\right) &= 1 \\ &\quad + z^{1/4}\sqrt{2}\,\sqrt{\pi}\,e^{2\sqrt{z}} \\ &\qquad \times \mathrm{erf}\left(\sqrt{2}\,z^{1/4}\right) \\ &\quad - z^{1/4}\sqrt{2}\,\sqrt{\pi}\,e^{-2\sqrt{z}}\,\mathrm{erfi}\left(\sqrt{2}\,z^{1/4}\right) \\ &\quad - 2\sqrt{z}\,\pi\,\mathrm{erf}\left(\sqrt{2}\,z^{1/4}\right) \\ &\qquad \times \mathrm{erfi}\left(\sqrt{2}\,z^{1/4}\right)\end{aligned}$$

$$\begin{aligned}\mathrm{F}\left(;1,\frac{5}{2},4;z\right) &= -\frac{18}{z^2}\,\mathrm{ber}_0\left(2\sqrt{2}\,z^{1/4}\right)^2 \\ &\quad - \frac{36-36\sqrt{z}+9\,z}{2\,z^{9/4}}\,\mathrm{ber}_0\left(2\sqrt{2}\,z^{1/4}\right) \\ &\qquad \times \mathrm{ber}_1\left(2\sqrt{2}\,z^{1/4}\right) \\ &\quad + \frac{36+36\sqrt{z}+9\,z}{2\,z^{9/4}}\,\mathrm{ber}_0\left(2\sqrt{2}\,z^{1/4}\right) \\ &\qquad \times \mathrm{bei}_1\left(2\sqrt{2}\,z^{1/4}\right) \\ &\quad - \frac{18+27\,z}{2\,z^{5/2}}\,\mathrm{ber}_1\left(2\sqrt{2}\,z^{1/4}\right)^2 \\ &\quad - \frac{36+36\sqrt{z}+9\,z}{2\,z^{9/4}}\,\mathrm{ber}_1\left(2\sqrt{2}\,z^{1/4}\right) \\ &\qquad \times \mathrm{bei}_0\left(2\sqrt{2}\,z^{1/4}\right) \\ &\quad - \frac{18}{z^2}\,\mathrm{bei}_0\left(2\sqrt{2}\,z^{1/4}\right)^2 \\ &\quad - \frac{36-36\sqrt{z}+9\,z}{2\,z^{9/4}}\,\mathrm{bei}_0\left(2\sqrt{2}\,z^{1/4}\right) \\ &\qquad \times \mathrm{bei}_1\left(2\sqrt{2}\,z^{1/4}\right) \\ &\quad - \frac{18+27\,z}{2\,z^{5/2}}\,\mathrm{bei}_1\left(2\sqrt{2}\,z^{1/4}\right)^2\end{aligned}$$

$$\mathrm{F}\left(\frac{3}{2};\frac{5}{2},5;z\right)$$
$$= -\frac{432 - 24\,z + 96\,z^2}{5\,z^3}\,\mathrm{I}_0\left(2\sqrt{z}\right) + \frac{432 + 192\,z + 48\,z^2}{5\,z^{7/2}}\,\mathrm{I}_1\left(2\sqrt{z}\right) - \frac{48}{5\,z}\,\pi \times \left(\mathrm{I}_0\left(2\sqrt{z}\right)\mathrm{L}_1\left(2\sqrt{z}\right) - \mathrm{I}_1\left(2\sqrt{z}\right)\mathrm{L}_0\left(2\sqrt{z}\right)\right)$$

$$\mathrm{F}\left(-\frac{1}{2},1,2;3,4;z\right)$$
$$= -\frac{480 + 3472\,z - 2100\,z^2}{525\,z^3} + \frac{480 + 3712\,z - 1024\,z^2 + 192\,z^3}{525\,z^3}\sqrt{1-z} - \frac{32}{5\,z^2}\log\left(\frac{1}{2} + \frac{\sqrt{1-z}}{2}\right)$$

$$\mathrm{F}\left(-\frac{1}{2},\frac{1}{2},1;\frac{3}{2},\frac{5}{2};z\right)$$
$$= -\frac{3 - 3\,z}{16\,z} - \frac{3 - 3\,z^2}{32\,z^{3/2}}\left(\log\left(1-\sqrt{z}\right) - \log\left(1+\sqrt{z}\right)\right) + \frac{3}{8\sqrt{z}}\,\mathrm{Li}_2\left(\sqrt{z}\right) - \frac{3}{8\sqrt{z}}\,\mathrm{Li}_2\left(-\sqrt{z}\right)$$

$$\mathrm{F}\left(1,2,3;\frac{1}{2},4;z\right)$$
$$= -\frac{45 - 30\,z - 3\,z^2}{4\,z^2\,(1-z)} - \frac{45 - 60\,z + 9\,z^2}{2\,z^{5/2}\,(1-z)^{3/2}}\sin^{-1}\left(\sqrt{z}\right) + \frac{45}{4\,z^3}\sin^{-1}\left(\sqrt{z}\right)^2$$

$$\mathrm{F}\left(\frac{1}{3};-\frac{2}{3},-\frac{1}{2},\frac{1}{2},1;z\right)$$
$$= \frac{1 + 3\sqrt{z}}{2}\,\mathrm{I}_0\left(4\,z^{1/4}\right) + \frac{1 - 3\sqrt{z}}{2}\,\mathrm{J}_0\left(4\,z^{1/4}\right) - \frac{7\,z^{1/4}}{4}\,\mathrm{I}_1\left(4\,z^{1/4}\right) + \frac{7\,z^{1/4}}{4}\,\mathrm{J}_1\left(4\,z^{1/4}\right)$$

$$\mathrm{F}\left(\frac{1}{2};-\frac{1}{3},-\frac{1}{2},-\frac{2}{3};z\right)$$
$$= \frac{2 - 6\,z^{1/3} + 9\,z^{2/3} - 18\,z}{6}\,e^{3\,z^{1/3}} + \frac{4 + 6\,z^{1/3} - 9\,z^{2/3} - 36\,z}{6}\,e^{-3\,z^{1/3}/2} \times \cos\left(\frac{3\sqrt{3}\,z^{1/3}}{2}\right) + \frac{z^{1/3}\left(2 + 3\,z^{1/3}\right)}{2}\sqrt{3}\,e^{-3\,z^{1/3}/2} \times \sin\left(\frac{3\sqrt{3}\,z^{1/3}}{2}\right)$$

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