Meijer G Function Representations

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Abstract

An algorithm for computing formula representations of instances of the Meijer G function is discussed. This algorithm is a generalization of an algorithm from a previous paper by the same author. The current paper discusses the Meijer G function briefly; the theory and lookup routine certificates of the new algorithm; and applications to the problem of definite integration.

1 Introduction

Our previous paper “Hypergeometric Function Representations” [15] presented an algorithm for computing formula representations of the hypergeometric function $F$ defined by

$$F\left(\mathbf{a}; \mathbf{b}; z\right) = \sum_{j=0}^{\infty} \frac{(\mathbf{a})_j z^j}{(\mathbf{b})_j j!} = \sum_{j=0}^{\infty} \Gamma\left(\frac{\mathbf{a} + j}{\mathbf{b}}, \frac{\mathbf{b}}{\mathbf{j} + 1 + j}\right) z^j$$

where we use notation

$$(\mathbf{a})_j = a (a + 1) (a + 2) \cdots (a + j - 1)$$

$$(\mathbf{a}_1, \ldots, \mathbf{a}_m)_j = (\mathbf{a}_1)_j \cdots (\mathbf{a}_m)_j$$

$$\Gamma\left(\mathbf{a}_1, \ldots, \mathbf{a}_m; \mathbf{b}_1, \ldots, \mathbf{b}_n\right) = \prod_{i=1}^{n} \Gamma(\mathbf{a}_i)$$

For example

$$F\left(-\frac{3}{2}, -\frac{1}{2}; \frac{1}{2}; z\right) = 2 + \frac{z}{\sqrt{1-z}} + 3 \frac{\sqrt{z}}{2} \sin^{-1}\left(\sqrt{z}\right)$$

is a typical formula representation. Ability to compute such representations is applicable to integration, differential equations, closed form summation, and difference equations [7][10][13].

The Meijer G function $G(\mathbf{a}; \mathbf{b}; \mathbf{c}; \mathbf{d}; z)$ is defined in the next section. It is a generalization of the hypergeometric function $F(\mathbf{a}; \mathbf{b}; z)$. Every hypergeometric function is a G function:

$$F\left(\mathbf{a}; \mathbf{b}; z\right) = \Gamma\left(\frac{\mathbf{b}}{\mathbf{a}}, 1 - \mathbf{a}; 0; 1 - \mathbf{b}; \log(-z)\right)$$

However, not every G function has a simple representation in terms of hypergeometric functions. In particular, Bessel functions $Y_\nu$ and $K_\nu (\mu \in \mathbb{Z})$, the Kelvin functions $\text{ker}_\mu$ and $\text{kei}_\mu (\mu \in \mathbb{Z})$, the Whittaker function $W_{\mu, \nu} (\nu \in \mathbb{Z})$, the Lommel function $S_{\mu, \nu}$, and the Legendre function $Q_{\nu}^\mu (\nu \in \mathbb{Z})$ can only be represented by G functions.

Our new algorithm computes formula representations such as

$$G\left(1; \frac{\mu}{2} + \frac{n}{2}, \frac{1}{2}; 0; \frac{\mu}{2} + \frac{n}{2}, \frac{1}{2}; -\log\left(\frac{z}{2}\right)\right)$$

$$= \frac{z^{n+1}}{\mu + n + 1} J_\mu(z) + z J_{\mu+1}(z) s_{n, \mu}(z)$$

$$- \frac{z}{\mu + n + 1} J_\mu(z) s_{n+1, \mu+1}(z)$$

An ability to produce such representations is crucially important to the solution of hypergeometric type integrals which appear copiously in various integral tables [5][11][12][13] used by scientists and mathematicians.

In this paper, we repeat some familiar themes from our previous work [15] of shift operator, contiguity relations, inversion, shift operator, suitable origins, accessible origins, proper sequences, and lookup certificates but in a new and different context. Just the same, the current paper is completely self-contained and will stand on its own.

2 Definition

We define the Meijer G function by the inverse Laplace transform
\[ G \left( \alpha; \beta; \gamma; \delta \right) = \frac{1}{2\pi i} \int_L \Gamma \left( \frac{1 - \alpha + y, \gamma - y}{\beta - y, 1 - \delta + y} \right) e^{y \gamma} dy \]

where \( L \) is one of three types of integration paths \( L_{\infty} \), \( L_{\infty}' \), and \( L_{\infty}'' \).

A schematic plot of the integration path \( L \) is shown below.

\[ \gamma + i \infty \]

\[ -\infty + i \phi_2 \]

\[ \infty + i \phi_2 \]

\[ \gamma - i \infty \]

\[ -\infty + i \phi_1 \]

\[ \infty + i \phi_1 \]

Contour \( L \) is one of three types of integration paths \( L_{\infty} \), \( L_{\infty}' \), and \( L_{\infty}'' \). Contour \( L_{\infty} \) starts at \( \infty + i \phi_1 \) and finishes at \( \infty + i \phi_2 \), Contour \( L_{\infty}' \) starts at \( -\infty + i \phi_1 \) and finishes at \( -\infty + i \phi_2 \), Contour \( L_{\infty}'' \) starts at \( \gamma - i \infty \) and finishes at \( \gamma + i \infty \). All the paths \( L_{\infty} \), \( L_{\infty}' \), and \( L_{\infty}'' \) put all \( c_j + k \) poles on the right and all other poles of the integrand (which must be of the form \(-1 + a_j + k\)) on the left. Define \( G_{\infty} \), \( G_{\infty}' \), and \( G_{\infty}'' \) to be the \( G \) functions defined by the \( L_{\infty} \), \( L_{\infty}' \), and \( L_{\infty}'' \) contours.

Theorem. \( G_{\infty} \) converges absolutely if

(1) \( \delta < 0 \) or

(2) \( \delta = 0 \) and \( \text{Re}(z) > 0 \) or

(3) \( \delta = 0 \), \( \text{Re}(z) = 0 \), and \( -\text{Re}(\sigma) < -1 \)

Theorem. \( G_{\infty} \) converges absolutely if

(1) \( \delta > 0 \) or

(2) \( \delta = 0 \) and \( \text{Re}(z) < 0 \) or

(3) \( \delta = 0 \), \( \text{Re}(z) = 0 \), and \( -\text{Re}(\sigma) < -1 \)

3 Relation to Traditional Notation

The Meijer \( G \) function is traditionally defined by an inverse Mellin transform

\[ G_{pq}^{mn} \left( z \right) = \frac{1}{2\pi i} \int_L \Gamma \left( \frac{1 - \alpha + y, \gamma - y}{\beta - y, 1 - \delta + y} \right) z^{-y} dy \]

Hence the traditional definition is related to our definition by

\[ G_{pq}^{mn} \left( z \right) = \frac{1}{2\pi i} \int_L \Gamma \left( \frac{1 - \alpha + y, \gamma - y}{\beta - y, 1 - \delta + y} \right) e^{y \gamma} dy \]

The new notation has some advantages over the old notation. First, the parameters of the Meijer \( G \) function are separated out into four natural groups: \( \alpha \), \( \beta \), \( \gamma \), and \( \delta \). Second, possibly more controversial, we place \( e^{y \gamma} \) instead of \( z^\gamma \) inside the integrand. We deem this desirable because of the “multi-valued” character of \( z^\gamma \). Finally, the \( pq \) subscripts and superscripts which are now redundant are omitted.

4 Properties

The Meijer \( G \) function has various properties [4] \( \Gamma \) [6] \( \Gamma \) [13]. Among those of interest is the following:

Theorem. (Basic Properties.)

\[ G \left( \mu, \nu; \beta; \gamma; \alpha, \delta \right) z = G \left( \beta; \gamma; \alpha, \delta \right) \]

\[ G \left( \mu; \nu; \alpha, \delta \right) z = G \left( \beta; \gamma; \alpha, \delta \right) \]

\[ e^{\gamma z} G \left( \alpha; \beta; \gamma; \delta \right) = G \left( \alpha; \beta; \gamma; \delta \right) \]

\[ e^{\gamma z} G \left( \alpha; \beta; \gamma; \delta \right) = G \left( \alpha; \beta; \gamma; \delta \right) \]
Theorem. (Duplication Formula.)
\[
G \left( a; \bar{b}; \bar{c}; \bar{d}; z \right) = \left( \frac{2 \pi}{k} \right)^{(k-1)/2} k^{1 + \sigma/2 - \pi} \times G \left( \Delta \left( \bar{a}, k \right), \Delta \left( \bar{b}, k \right), \Delta \left( \bar{c}, k \right), \Delta \left( \bar{d}, k \right); \right.
\]
\[
z + k \delta \log(\bar{k}) \Bigg)\]

where we use notation
\[
\Delta \left( \bar{a}, k \right) = \frac{\bar{a}}{k}, \frac{\bar{a} + 1}{k}, \frac{\bar{a} + 2}{k}, \ldots, \frac{\bar{a} + k - 1}{k}
\]

Theorem. (Slater's Theorem.) If \( G_\infty \) converges and the elements of \( \bar{c} \) are distinct mod 1, then
\[
G \left( a; \bar{b}; \bar{c}; \bar{d}; z \right) = \sum_{n=1}^{\infty} \left( \prod_{n=1}^{n} \left( 1 - \bar{a} + c + \bar{c}^* - c_n \right) \right) e^{c \cdot z}
\]
\[
\times F \left( 1 - \bar{a} + c, 1 - \bar{b} + c; 1 - \bar{c}^* + c, 1 - \bar{d} + c; \right)
\]
\[
(-1)^{n-1} c^n \right)
\]

where \( c^* = \bar{c} \) with \( c_n \) omitted.

5 Integration Theorems

Four theorems below are not original but serve as a small reference guide to the reader indicating the usefulness of the Meijer G function to solving integration problems. These theorems are very general since many special functions can be represented as G functions. We omit some rather complicated technical conditions on parameters which appear in the last three theorems pertaining to definite integration. Readers may consult section 2.24 of Integrals and Series Volume 3: More Special Functions [13] for their complete statement and additional theorems.

Theorem. (Indefinite Integration.)
\[
\int G \left( a; \bar{b}; \bar{c}; \bar{d}; z \right) dz = G \left( 1, \bar{a}; \bar{b}; \bar{c}; 0, \bar{d}; z \right)
\]

Theorem. (One G Function.)
\[
\int_0^\infty z^n G \left( a; \bar{b}; \bar{c}; \bar{d}; u \log(z) + v \right) dz = \frac{1}{u} e^{-\alpha v} \Gamma \left( -\alpha + 1, \bar{c}^* + \alpha \right)
\]

where
\[
\alpha = \frac{t + 1}{u}
\]

Theorem. (Two G Functions.)
\[
\int_0^\infty z^n G_1 G_2 dz = \frac{1}{u} e^{-\alpha v} G_3
\]

where
\[
G_1 = G \left( a_1; \bar{b}_1; \bar{c}_1; \bar{d}_1; u \log(z) + v_1 \right)
\]
\[
G_2 = G \left( a_2; \bar{b}_2; \bar{c}_2; \bar{d}_2; u \log(z) + v_2 \right)
\]
\[
G_3 = G \left( a_1, -c^*_2 - \alpha + 1; \bar{b}_1, -d^*_2 - \alpha + 1; \right)
\]
\[
\alpha = \frac{t + 1}{u}
\]

Theorem. (Cauchy Principal Value Integral.)
\[
\int_0^\infty G \left( a; \bar{b}; \bar{c}; \bar{d}; \log(z) + v \right) \frac{dz}{z - \mu}
\]
\[
= -\pi G \left( 0, \bar{a}; -\frac{1}{2}, \bar{b}; 0, \bar{c}; -\frac{1}{2}, \bar{d}; v + \log(\mu) \right)
\]

6 Shift Operators

Define the shift operators \( A_i \Gamma B_i \Gamma C \Gamma \Gamma D_i \) and \( D_i \) by
\[
A_i = D + (-a_i + 1)
\]
\[
B_i = D + (b_i - 1)
\]
\[
C_i = D + c_i
\]
\[
D_i = D - d_i
\]

where \( D = (\partial/\partial z) \) is the operator for differentiation. It can be seen that \( A_i \) and \( B_i \) decrement indices and that \( C_i \) and \( D_i \) increment indices. Visibly
\[
A_i G \left( a; \bar{b}; \bar{c}; \bar{d}; z \right) = G \left( a - c_i; \bar{b}; \bar{c}; \bar{d}; z \right)
\]
\[
B_i G \left( a; \bar{b}; \bar{c}; \bar{d}; z \right) = G \left( a; \bar{b} - c_i; \bar{c}; \bar{d}; z \right)
\]
\[
C_i G \left( a; \bar{b}; \bar{c}; \bar{d}; z \right) = G \left( a; \bar{b}; \bar{c} + c_i; \bar{d}; z \right)
\]
\[
D_i G \left( a; \bar{b}; \bar{c}; \bar{d}; z \right) = G \left( a; \bar{b}; \bar{c} + 1; \bar{d} + 1; z \right)
\]

where \( c \) are unit vectors.

7 Differential Equation

Applying products of shift operators to \( G \left( a; \bar{b}; \bar{c}; \bar{d}; z \right) \) we see that
\[
\prod_{i=1}^{m} A_i \prod_{i=1}^{n} B_i \prod_{i=1}^{p} C_i \prod_{i=1}^{q} D_i G \left( a; \bar{b}; \bar{c}; \bar{d}; z \right) = G \left( a; \bar{b} - 1; \bar{c}; \bar{d}; z \right)
\]
\[
\prod_{i=1}^{p} C_i \prod_{i=1}^{q} D_i G \left( a; \bar{b}; \bar{c} + 1; \bar{d} + 1; z \right)
\]
It can be checked that
\[ e^z G \left( \bar{a} - 1; \bar{b} - 1; \bar{c}; \bar{d}; z \right) = G \left( \bar{a}; \bar{b}; \bar{c} + 1; \bar{d} + 1; z \right) \]

Hence,
\[
\left( e^z \sum_{i=1}^{m} A_i \prod_{i=1}^{n} B_i - \sum_{i=1}^{p} C_i \prod_{i=1}^{q} D_i \right) G \left( \bar{a}; \bar{b}; \bar{c}; \bar{d}; z \right) = 0
\]

Converting to D notation, we get the differential equation for \( G \left( \bar{a}; \bar{b}; \bar{c}; \bar{d}; z \right) \). If we let \( L_1 \Gamma L_2 \Gamma \Lambda \) and \( L \) be the operators
\[
L_1 = (-1)^{m+p} e^z \prod_{j=1}^{m} (D + (-a_j + 1)) \prod_{j=1}^{n} (D + (1 - b_j))
\]

\[
L_2 = \prod_{j=1}^{p} (D - c_j) \prod_{j=1}^{q} (D - d_j)
\]

\[
L = L_1 - L_2
\]

then the differential equation for \( G \left( \bar{a}; \bar{b}; \bar{c}; \bar{d}; z \right) \) can be written
\[
L G \left( \bar{a}; \bar{b}; \bar{c}; \bar{d}; z \right) = 0
\]

8 Contiguity Relations

Operator \( L \) is a polynomial in \( D \) but
\[
D + \mu = A_i + (\mu + a_i - 1)
\]

\[
D + \mu = -B_i + (\mu + b_i - 1)
\]

\[
D + \mu = -C_i + (\mu + c_i)
\]

\[
D + \mu = D_i + (\mu + d_i)
\]

so \( L \) can also be expressed as a polynomial in terms of shift operators \( A_i \Gamma B_i \Gamma C_i \Gamma \Lambda \) and \( D \), converting the differential equation for \( G \left( \bar{a}; \bar{b}; \bar{c}; \bar{d}; z \right) \) into a difference equation among contiguous instances of \( G \) which we call a contiguity relation.

Let \( X \) stand for \( A \Gamma B \Gamma C \Gamma \Lambda \) for \( D \) and \( \chi \) stand for \( \alpha \Gamma \beta \Gamma \gamma \Gamma \delta \) respectively. If we express \( L \) as a polynomial in \( X \), then we get
\[
L_1 = (\pm) e^z X_i^{m+n} + \ldots + 0
\]

\[
L_2 = (\pm) X_i^{m+q} + \ldots + \chi_0 \left( \bar{a}, \bar{b}, \bar{c}, \bar{d}, z \right)
\]

\[
L = \chi_0 \left( \bar{a}, \bar{b}, \bar{c}, \bar{d}, z \right) X_i^q + \ldots + \chi_0 \left( \bar{a}, \bar{b}, \bar{c}, \bar{d}, z \right)
\]

where the \( \pm \) signs depend on \( m \Gamma n \Gamma p \Gamma q \) and whether \( X \) is \( A \Gamma B \Gamma C \Gamma D \) and \( d = \max(m + n, p + q) \).

These results let us define
\[
A_i^{-1} = -\sum_{j=0}^{d-1} \frac{a_{j+1}}{a_0} \left( \bar{a} + \bar{e}_j, \bar{b}, \bar{c}, \bar{d}, z \right) A_i^j
\]

\[
B_i^{-1} = -\sum_{j=0}^{d-1} \frac{b_{j+1}}{b_0} \left( \bar{a}, \bar{b} + \bar{e}_j, \bar{c}, \bar{d}, z \right) B_i^j
\]

\[
C_i^{-1} = -\sum_{j=0}^{d-1} \frac{c_{j+1}}{c_0} \left( \bar{a}, \bar{b}, \bar{c} - \bar{e}_j, \bar{d}, z \right) C_i^j
\]

\[
D_i^{-1} = -\sum_{j=0}^{d-1} \frac{d_{j+1}}{d_0} \left( \bar{a}, \bar{b}, \bar{c}, \bar{d} - \bar{e}_j, z \right) D_i^j
\]

The coefficients of these polynomials in \( A_i \Gamma B_i \Gamma C_i \Gamma \Lambda \) and \( D \) are defined when
\[
\alpha_0 \left( \bar{a} + \bar{e}_i, \bar{b}, \bar{c}, \bar{d}, z \right)
\]

\[
= -\prod_{j=1}^{p} (c_j + a_i) \prod_{j=1}^{q} (d_j + a_i) \neq 0
\]

\[
\beta_0 \left( \bar{a}, \bar{b} + \bar{e}_i, \bar{c}, \bar{d}, z \right)
\]

\[
= -\prod_{j=1}^{p} (c_j + b_i) \prod_{j=1}^{q} (d_j + b_i) \neq 0
\]

\[
\gamma_0 \left( \bar{a}, \bar{b}, \bar{c} - \bar{e}_i, \bar{d}, z \right)
\]

\[
= \left( -1 \right)^{m+n} \prod_{j=1}^{m} (-a_j + c_i) \prod_{j=1}^{n} (-b_j + c_i) e^z \neq 0
\]

\[
\delta_0 \left( \bar{a}, \bar{b}, \bar{c}, \bar{d} - \bar{e}_i, z \right)
\]

\[
= \left( -1 \right)^{m+n} \prod_{j=1}^{m} (-a_j + d_i) \prod_{j=1}^{n} (-b_j + d_i) e^z \neq 0
\]

9 Contiguity Relations II

For example, using the ideas of the previous sections our routine Contigu computes the following contiguity relation:
\[
G(a_1 + 1; c_1, c_2; d_1; z) = \frac{1}{(a_1 - d_1)} \frac{1}{(a_1 - c_2)} \frac{1}{(a_1 - c_1)} G(a_1 - 2; c_1, c_2; d_1; z) \\
- \frac{3 a_1 - c_1 - c_2 - d_1 - 3}{(a_1 - d_1)} \frac{1}{(a_1 - c_2)} \frac{1}{(a_1 - c_1)} G(a_1 - 1; c_1, c_2; d_1; z) \\
- \left(3a_1^2 - 2a_1c_1 - 2a_1c_2 - 2a_1d_1 + c_1c_2 + c_1d_1 + c_2d_1 - e^z - 3a_1 + c_1 + c_2 + d_1 + 1 \right) \\
\times (a_1 - d_1)^{-1} (a_1 - c_2)^{-1} (a_1 - c_1)^{-1} \times G(a_1; c_1, c_2; d_1; z)
\]

10 Proper Sequences and Suitable Origins

Definition A sequence \( S \) of shift and inverse shift operators \( A_1 \Gamma B_1 \Gamma C_1 \Gamma D_1 \Gamma A_1^{-1} \Gamma B_1^{-1} \Gamma C_1^{-1} \Gamma \text{and } D_1^{-1} \) is a proper sequence if the composition \( S \) is defined.

Definition A quadruple \( (a_0; b_0; c_0; d_0) \) is a suitable origin if \( \{a_0, b_0\} \) and \( \{c_0, d_0\} \) are disjoint. Hence \( a_0 \neq c_0, a_0 \neq d_0, b_0 \neq c_0, b_0 \neq d_0 \).

Definition A quadruple \( (a; b; c; d) \) is accessible from a quadruple \( (a_0; b_0; c_0; d_0) \) if there exists a constant \( t \in \mathbb{C} \) and a proper sequence \( S \) of shift and inverse shift operators \( A_1 \Gamma B_1 \Gamma C_1 \Gamma D_1 \Gamma A_1^{-1} \Gamma B_1^{-1} \Gamma C_1^{-1} \Gamma \text{and } D_1^{-1} \) such that

\[
G(a; b; c; d; z) = S_1 \ldots S_1 G(a_0 + t; b_0 + t; c_0 + t; d_0 + t; z)
\]

11 Strategy

Assume \( \{a, b\} \) and \( \{c, d\} \) are disjoint. Suppose \( t \in \mathbb{C} \) and \( \vec{k} = \vec{a_0} + t - \vec{a} \Pi = \vec{b_0} + t - \vec{b} \Pi \vec{m} = \vec{c} - \vec{c_0} - t \Gamma \vec{n} = \vec{d} - \vec{d_0} - t \in \mathbb{Z} \). We would try

\[
G(a; b; c; d; z) = \prod_{i=1}^{m} A_i \prod_{i=1}^{n} B_i \prod_{i=1}^{p} C_i^{m_i} \prod_{i=1}^{q} D_i^{n_i} e^z \\
\times G(a_0; b_0; c_0; d_0; z)
\]

but this will not always work because of restrictions on where \( A_1^{-1} \Gamma B_1^{-1} \Gamma C_1^{-1} \Gamma \text{and } D_1^{-1} \) are defined.

Given any vector \( \vec{v} \) let \( \pi(\vec{v}) = (v_{\pi(1)}, \ldots, v_{\pi(|\vec{v}|)}) \).

Let \( \vec{x} = (a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p, d_1, \ldots, d_l) \)

Let \( \pi \) be a permutation which sorts \( \vec{x} \) into non-descending order. Let \( \vec{y} = \pi(\vec{x}) \). Then \( [\vec{y}]_r \) is non-descending for every \( r \in [0,1) \).

Assume \( (a_0; b_0; c_0; d_0) \) is a suitable origin such that for every \( t \in \mathbb{C} \) and \( \vec{k} = \vec{a_0} + t - \vec{a} \Pi = \vec{b_0} + t - \vec{b} \Pi \vec{m} = \vec{c} - \vec{c_0} - t \Gamma \vec{n} = \vec{d} - \vec{d_0} - t \in \mathbb{Z} \).

\[
\vec{x}_0 = (a_0, \ldots, a_m, b_0, \ldots, b_n, c_0, \ldots, c_p, d_0, \ldots, d_l)
\]

\[
\vec{X} = (A_1^{k_1}, \ldots, A_m^{k_m}, B_1^{l_1}, \ldots, B_n^{l_n}, C_1^{m_1}, \ldots, C_p^{m_p}, D_1^{n_1}, \ldots, D_l^{n_l})
\]

Let \( \vec{y}_0 = \pi(\vec{x}_0) \) and \( \vec{Y} = \pi(\vec{X}) \). Assume \( [\vec{y}_0]_r \) is non-descending for every \( r \in [0,1) \).

For any given \( r \in [0,1) \) plot the elements of \( [\vec{y}]_r \) and \( [\vec{y}_0]_r \) as a function of position. Call the resulting monotonic polygonal curves \( Y \) and \( Y_0 \). For example I've might get this picture:

![Graphical representation of \( Y \) and \( Y_0 \)](image-url)
12 Formula Algorithm

\textbf{proc} Formula(\(\vec{a}, \vec{b}, \vec{c}, \vec{d}\))
\begin{align*}
\vec{a}:&=\text{sort}(\vec{a}) \\
\vec{b}:&=\text{sort}(\vec{b}) \\
\vec{c}:&=\text{sort}(\vec{c}) \\
\vec{d}:&=\text{sort}(\vec{d}) \\
\end{align*}
Delete elements \(\vec{a}\) and \(\vec{d}\) have in common.
Delete elements \(\vec{b}\) and \(\vec{c}\) have in common.
\begin{align*}
[\vec{a}_0, \vec{b}_0, \vec{c}_0, \vec{d}_0, B, C, M, \rho] :=& \text{Lookup}(\vec{a}, \vec{b}, \vec{c}, \vec{d}); \\
[\vec{a}_0, \vec{b}_0, \vec{c}_0, \vec{d}_0, \text{plan}] :=& \text{Plan}(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{a}_0, \vec{b}_0, \vec{c}_0, \vec{d}_0); \\
\text{for} \ bucket \ in \ \text{plan} \ \text{do} \\
     \left[\text{shift}, e\right] := \text{bucket}; \\
\text{if} \ e < 0 \ \text{then} \\
     \text{for} \ j \ \text{from} \ 1 \ \text{to} \ -e \ \text{do} \\
         [\vec{a}_0, \vec{b}_0, \vec{c}_0, \vec{d}_0, C] := \\
         \text{Unshift}(\text{shift}, \vec{a}_0, \vec{b}_0, \vec{c}_0, \vec{d}_0, z^{\rho}, C, M); \\
     \text{od}; \\
\text{elseif} \ e > 0 \ \text{then} \\
     \text{for} \ j \ \text{from} \ 1 \ \text{to} \ e \ \text{do} \\
         [\vec{a}_0, \vec{b}_0, \vec{c}_0, \vec{d}_0, C] := \\
         \text{Shift}(\text{shift}, \vec{a}_0, \vec{b}_0, \vec{c}_0, \vec{d}_0, z^{\rho}, C, M); \\
     \text{od}; \\
\text{fi;} \\
\text{od;} \\
\text{return} \ \text{subs}(z = z^{1/\rho}, C \cdot B); \\
\end{align*}

13 Lookup Routine

The \textbf{Lookup} routine currently consists of 48 procedures each of which will effect infinitely many \([\vec{a}_0, \vec{b}_0, \vec{c}_0, \vec{d}_0, B, C, M, \rho] \text{ certificates to a virtual lookup table.}

The following table summarizes the number of formulas in \textbf{Lookup} by their \((m, n, p, q)\) classification:

<table>
<thead>
<tr>
<th>((m, n, p, q))</th>
<th>#</th>
<th>((0, 0, 2, 2))</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 3, 1))</td>
<td>2</td>
<td>((0, 1, 2, 0))</td>
<td>1</td>
</tr>
<tr>
<td>((0, 1, 2, 1))</td>
<td>3</td>
<td>((0, 1, 2, 3))</td>
<td>1</td>
</tr>
<tr>
<td>((0, 1, 3, 0))</td>
<td>1</td>
<td>((0, 1, 3, 2))</td>
<td>1</td>
</tr>
<tr>
<td>((0, 1, 4, 1))</td>
<td>1</td>
<td>((0, 2, 3, 1))</td>
<td>2</td>
</tr>
<tr>
<td>((0, 2, 4, 0))</td>
<td>1</td>
<td>((0, 3, 4, 1))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 0, 1, 2))</td>
<td>3</td>
<td>((1, 0, 2, 0))</td>
<td>2</td>
</tr>
<tr>
<td>((1, 0, 2, 1))</td>
<td>6</td>
<td>((1, 0, 3, 0))</td>
<td>3</td>
</tr>
<tr>
<td>((1, 1, 2, 1))</td>
<td>1</td>
<td>((1, 1, 2, 2))</td>
<td>3</td>
</tr>
<tr>
<td>((1, 1, 3, 1))</td>
<td>2</td>
<td>((1, 1, 4, 0))</td>
<td>1</td>
</tr>
<tr>
<td>((2, 0, 2, 2))</td>
<td>2</td>
<td>((2, 0, 3, 1))</td>
<td>2</td>
</tr>
<tr>
<td>((2, 1, 2, 3))</td>
<td>2</td>
<td>#</td>
<td>1</td>
</tr>
</tbody>
</table>

14 Results and Conclusion

Due to their complexity and lack of space we will not present a number of more advanced theorems related to calculation of Meijer G Function Representations. We just say that these theorems go by names such as \textbf{Paired Index Theorems} (similar to theorems in Adamchik [3]) and \textbf{PFD Duplication Formula} (related to a similar formula in Roach [15]) and an \textbf{Expansion Theorem} (a generalization of Slater’s Theorem).

One of our long term goals is to enlarge our \textbf{Lookup} routine to the point that our algorithm should basically reproduce nearly all 879 (roughly) of the formula representations for the Meijer G function listed in chapter 8 of \textit{Integrals and Series Volume 3: More Special Functions} [13]. We are not at that point yet but progress is good. Every formula in this book through our algorithm turns into infinitely many formulas. We also envision that our algorithm will appear as an important subroutine inside general routines which solve integration problems.

In the course of this work we discovered mistakes in formulas 2(19)\(\Gamma\)2(7)\(\Gamma\)12(8)\(\Gamma\)15(7)\(\Gamma\)15(8)\(\Gamma\)20(8)\(\Gamma\)20(45)\(\Gamma\)22(15)\(\Gamma\)22(16)\(\Gamma\)22(21)\(\Gamma\)22(22)\(\Gamma\)22(23)\(\Gamma\)23(34)\(\Gamma\)25(5)\(\Gamma\)29(15)\(\Gamma\)29(16)\(\Gamma\)40(6)\(\Gamma\)40(22)\(\Gamma\)40(23)\(\Gamma\)49(41)\(\Gamma\)49(42)\(\Gamma\)49(44)\(\Gamma\)18(15)\(\Gamma\)18(16)\(\Gamma\)43(1)\(\Gamma\)43(2)\(\Gamma\)46(9)\(\Gamma\)46(10) of section 8.4 of \textit{Integrals and Series Volume 3: More Special Functions} [13]. We have not inspected sections 41 and 42 discussing the Legendre functions \(P^\mu_\nu\) and \(Q^\mu_\nu\) closely yet to comment about their correctness but otherwise this list of errors may be nearly comprehensive.

15 Gallery

The following integrals most of which appear in \textit{Integrals and Series} [11]\(\Gamma\)12\(\Gamma\)13 were calculated with the aid of the theorems and algorithm described in this paper. The performance of two different computer algebra systems on this test suite is as follows: Maple 5.4 was able to compute a formula for one integral and left all the other integrals unevaluated. Mathematica 2.2 left six integrals unevaluated it produced four answers which still contained hypergeometric functions \(\Gamma\) and only computed formulas for three of these integrals.

\[
\int z^n J_\mu(z) \, dz = \frac{\mu + n + 1}{\mu + n + 1} J_\mu(z) + z J_{\mu+1}(z) s_{n, \mu}(z) - \frac{z}{\mu + n + 1} J_\mu(z) s_{n+1, \mu+1}(z)
\]
\[ \int_0^\infty \frac{\sin(b x)}{(x^2 + x^2)^{3/2}} \, dx \]
\[ = -\csc(\pi \rho) \left( \frac{2^{1/2} - \rho^{1/2} \pi^{3/2}}{2 \Gamma(\rho)} \right) I_{2x^{3/2}}(b z) \]
\[ - \frac{2^{-1/2} - \rho \pi^{1/2} \rho^{1/2} \Gamma(-\rho + 1/2)}{\rho - 1} \cdot \left( 2 \rho + 1 \right) \csc(\pi \rho) I_{\rho + 1/2}(b z) \]
\[ - \frac{\pi^{3/2} \rho^{-3/2} 2^{1/2} - \rho + 1}{2 \Gamma(\rho)} \csc(\pi \rho) I_{\rho + 1/2}(b z) \]

\[ \int_0^\infty \frac{\cos(a x^2 + 2 b x)}{(x^2 + x^2)^{3/2}} \, dx \]
\[ \frac{\sqrt{\pi} \sqrt{2} \cos \left( \frac{b^2}{\pi} \right)}{4 \sqrt{a}} + \frac{\sqrt{\pi} \sqrt{2} \sin \left( \frac{b^2}{\pi} \right)}{4 \sqrt{a}} \]
\[ - \frac{\sqrt{\pi} \sqrt{2} \cos \left( \frac{b^2}{\pi} \right)}{2 \sqrt{a}} C \left( \frac{\sqrt{2} b}{\sqrt{\pi} \sqrt{a}} \right) \]
\[ - \frac{\sqrt{\pi} \sqrt{2} \sin \left( \frac{b^2}{\pi} \right)}{2 \sqrt{a}} S \left( \frac{\sqrt{2} b}{\sqrt{\pi} \sqrt{a}} \right) \]

\[ \int_0^\infty \cos(b x) \tan^{-1} \left( \frac{a}{x^2} \right) \, dx \]
\[ \frac{\pi \sin \left( \sqrt{\pi} \sqrt{a} \right)}{b} \exp \left( -\frac{\sqrt{2} b \sqrt{a}}{2} \right) \]

\[ \int_0^\infty \frac{\cos(b \sqrt{x}) J_0(b x)}{\sqrt{x}} \, dx \]
\[ \frac{b \pi}{4 c} J_0 \left( \frac{b^2}{8 c} \right)^2 + \frac{36 \pi c}{b^3} J_0 \left( \frac{b^2}{8 c} \right)^2 \]
\[ - \frac{b \pi}{4 c} J_0 \left( \frac{b^2}{8 c} \right)^2 - \frac{6 \pi}{b} J_0 \left( \frac{b^2}{8 c} \right) J_0 \left( \frac{b^2}{8 c} \right) \]

\[ \int_0^\infty J_1(b x) J_1(b x) \, dx \]
\[ \frac{2 b^2 - 2 c^2}{3 b \pi} K \left( \frac{b^2}{c} \right) + \frac{2 b^2 + 2 c^2}{3 b \pi} E \left( \frac{b^2}{c} \right) \]

\[ \int_0^\infty x^{3/2} J_0(b x) J_0(c x) \, dx \]
\[ = \frac{1}{2 \sqrt{b} \Gamma \left( \frac{3}{4} \right)^2 (b^2 - c^2)} K \left( \frac{2 - 2 \sqrt{b^2 - c^2}}{2} \right) \]
\[ + \frac{1}{b^{5/2} \Gamma \left( \frac{3}{4} \right)^2 (b^2 - c^2)^{3/2}} E \left( \frac{2 - 2 \sqrt{b^2 - c^2}}{2} \right) \]

\[ \int_0^\infty e^{i p x} H_0^{(1)}(c x) \, dx \]
\[ = \frac{1}{c \sqrt{c^2 - p^2}} \sin^{-1} \left( \frac{p}{c} \right) \]

\[ \int_0^\infty e^{-p x} I_1(c x) \, dx \]
\[ = -\frac{1}{2} + \frac{p^2}{2 \pi c^2} E \left( \frac{2 c}{p} \right) + \frac{4 c^2 - p^2}{2 \pi c^2} K \left( \frac{2 c}{p} \right) \]

\[ \int_0^\infty e^{-p x} I_1(c x)^2 \, dx \]
\[ = -\frac{1}{2} + \frac{p^2}{2 \pi c^2} E \left( \frac{2 c}{p} \right) + \frac{4 c^2 - p^2}{2 \pi c^2} K \left( \frac{2 c}{p} \right) \]

\[ \int_0^\infty x^{\nu+1} \sin \left( \frac{c x^2}{2 a} \right) K_\nu(c x) \, dx \]
\[ = \frac{2^\nu e^{-\nu-1} a (\nu-1) \Gamma(\nu-1)}{2} \]
\[ + \frac{\pi a^{\nu+1} \Gamma \left( \frac{\nu+1}{2} \right)}{4 c} \frac{\sin \left( \frac{\pi a^2}{4 c} \right)}{\sin \left( \frac{\pi a}{4 c} \right)} \]
\[ - \frac{\pi a^{\nu+1} \Gamma \left( \frac{\nu+1}{2} \right)}{4 c} \frac{\cos \left( \frac{\pi a}{4 c} \right)}{\cos \left( \frac{\pi a}{4 c} \right)} \]
\[ + \frac{\Gamma(\nu-1) a^{\nu+1/2} \sqrt{\frac{2}{c}}}{2 c^{3/2}} \frac{\Gamma \left( \frac{\nu-3/2}{2} \right)}{\Gamma(\nu-1)} \frac{\sin \left( \frac{\pi a^2}{4 c} \right)}{\sin \left( \frac{\pi a}{4 c} \right)} \]
\[ + \frac{\Gamma(\nu-1) a^{\nu+3/2} \sqrt{\frac{2}{c}}}{4 \sqrt{c}} \frac{\sin \left( \frac{\pi a}{4 c} \right)}{\sin \left( \frac{\pi a}{4 c} \right)} \]
\begin{align}
\int_0^\infty \sin (b x^2) K_\nu (c x) \, dx \\
&= -\frac{\pi^{3/2} \csc \left( \frac{\nu + 1}{4} \right) \sin \left( \frac{c^2}{8b} \right) \csc \left( \frac{\nu}{2} \right) J_{\nu} \left( \frac{c^2}{8b} \right)}{16 \sqrt{b}} \\
&\quad - \frac{\pi^{3/2} \sec \left( \frac{\nu + 1}{4} \right) \cos \left( \frac{c^2}{8b} \right) \csc \left( \frac{\nu}{2} \right) J_{-\nu} \left( \frac{c^2}{8b} \right)}{16 \sqrt{b}} \\
&\quad + \frac{\pi^{3/2} \sec \left( -\frac{\nu - 1}{4} \right) \sin \left( \frac{c^2}{8b} \right) \csc \left( \frac{\nu}{2} \right) J_{-\nu} \left( \frac{c^2}{8b} \right)}{16 \sqrt{b}} \\
&\quad + \frac{\pi^{3/2} \sec \left( -\frac{\nu - 1}{4} \right) \cos \left( \frac{c^2}{8b} \right) \csc \left( \frac{\nu}{2} \right) J_{\nu} \left( \frac{c^2}{8b} \right)}{16 \sqrt{b}}
\end{align}

\begin{align}
\int_0^\infty x J_{\nu} (b x) \left( Y_\nu (c x) - H_\nu (c x) \right) \, dx \\
&= -\frac{2 b^{\nu - 1} c^\nu b^2 \cos (\pi \nu)}{\pi \left( b^2 - c^2 \right)} + \frac{2 b^{3-\nu} c^{\nu+1} b^2 \cos (\pi \nu)}{\pi \left( b^2 - c^2 \right)}
\end{align}

References


